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Dynamics and thermodynamics of a simple model similar to self-gravitating systems: the HMF model

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Abstract. We discuss the dynamics and thermodynamics of the Hamiltonian Mean Field model (HMF) which is a prototypical system with long-range interactions. The HMF model can be seen as the one Fourier component of a one-dimensional self-gravitating system. Interestingly, it exhibits many features of real self-gravitating systems (violent relaxation, persistence of metaequilibrium states, slow collisional dynamics, phase transitions,...) while avoiding complicated problems posed by the singularity of the gravitational potential at short distances and by the absence of a large-scale confinement. We stress the deep analogy between the HMF model and self-gravitating systems by developing a complete parallel between these two systems. This allows us to apply many technics introduced in plasma physics and astrophysics to a new problem and to see how the results depend on the dimension of space and on the form of the potential of interaction. This comparative study brings new light in the statistical mechanics of self-gravitating systems. We also mention simple astrophysical applications of the HMF model in relation with the formation of bars in spiral galaxies.

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1 Introduction

The statistical mechanics of systems with long-range interactions is currently a topic of active research in physics because it differs in many respects from that of more familiar systems with short-range forces that are extensive [1]. Among long-range interactions, gravity is probably the most important and most fundamental example [2,3]. However, the statistical mechanics of self-gravitating systems initiated by Antonov [4] and Lynden-Bell [5] is complicated due to the divergence of the gravitational force at short distances and to the absence of shielding (or confinement) at large distances. These difficulties are specific to the gravitational force and not to the long-range nature of the interaction. Therefore, it may be of conceptual interest to consider simpler systems with long-range interactions to distinguish what is specific to the gravitational force and what is common to systems with long-range interactions.

A toy model of systems with long-range interactions is the so-called HMF (Hamiltonian Mean Field) model. It consists of N particles moving on a circle and interacting via a cosine binary potential. This can be seen as a one-dimensional plasma where the potential of interaction is truncated to one mode. This model is of great conceptual interest because it exhibits many features present in more realistic systems with long-range interactions such as gravitational systems. In addition, it is sufficiently simple to allow for accurate numerical simulations and analytical results.

To our knowledge, what is now called the HMF model was first introduced by Konishi & Kaneko [6]. They found that a cluster is formed in some cases and that the system remains uniform in other cases. Inagaki & Konishi [7] realized that the Konishi-Kaneko system is nothing but the one Fourier component of a one-dimensional self-gravitating system and explained the formation of clusters as an instability similar to the Jeans instability in self-gravitating systems described by the Vlasov equation. Inagaki [8] studied the thermodynamical stability of the Konishi-Kaneko system and identified the existence of a second order phase transition at a critical temperature T_c. Above T_c the only statistical equilibrium state is uniform, whereas below T_c this uniform state looses its thermodynamical stability and clustered states appear. In order to justify his results dynamically, Inagaki [9] developed a "collisional" kinetic theory of the Konishi-Kaneko system

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based on results of plasma physics and proposed to model the dynamics of the system by the Lenard-Balescu equation for a one dimensional plasma truncated to one mode. However, as we shall see, his conclusions demand further discussion.

The same model was considered at about the same time by Pichon and Lynden-Bell (Pichon [10]) who gave an astrophysical application of this model in relation with the formation of *bars* in galactic disks. In their approach, the stars follow rigid elliptical orbits with eccentricity e. If ϕ_i represents the inclination of ellipse i and Ω_i its angular velocity, the torque exerted by an orbit to the other can be written $\alpha^{-1}d\Omega_1/dt = \partial\psi_{12}/\partial\phi_1$ where α^{-1} is the adiabatic moment of inertia of the inner Lindblad orbit and $\psi_{12} = GA^2\cos 2(\phi_1-\phi_2)$ is the effective alignement potential. At high temperatures, the orbits are almost uniformly distributed in space and the system is in a disk phase (see Fig. 4). However, below some critical temperature T_c, the ellipses tend to align to each other and form a bar (see Fig. 5). Those bars are reported observationally in real galaxies. Pichon and Lynden-Bell studied the linear stability of these bars with respect to the Vlasov equation and proposed that the clustered phase could result from a process of violent relaxation, a concept introduced by Lynden-Bell [11] to explain the regularity of collisionless stellar systems such as elliptical galaxies.

The Konishi-Kaneko system, now called the HMF model, also appeared in statistical mechanics [12]. In that context, the motivation was to devise a simple model with long-range interactions keeping the richness of more realistic systems while being amenable to a full analytical and numerical treatement. Excitingly, this simple model displays a lot of interesting features (violent relaxation, persistence of metaequilibrium states, slow collisional relaxation, phase transitions,...) also present in other systems with long-range interactions such as stellar systems and 2D vortices [3]. The properties of the HMF model have been studied in great detail in a lot of recent papers (see Dauxois et al. [13] for a review). Despite its oversimplification, the HMF model can be seen as a pedagogical model to take a step into the physics of systems with long-range interactions. It is said sometimes to represent the "harmonic oscillator" of systems with long-range interactions. This probably explains its popularity.

In the present paper, we shall emphasize the connection between the HMF model and the results established in astrophysics and plasma physics. In particular, we will adapt the methods developed for 3D self-gravitating systems to the case of a one-dimensional system of particles with cosine interactions. The motivation of this extension is two-fold. The first is to show that the results obtained in astrophysics and plasma physics can have applications in other domains of physics, including the HMF model (this has not been sufficiently appreciated by workers in that field since the early work of Inagaki). The second is to stress the analogies and the differences which appear in long-range systems as we change the dimension of space and the potential of interaction. Among the analogies between 3D self-gravitating systems and the HMF model, we note: the concept of violent relaxation and the slow collisional dynamics. Among the differences, we note: the equivalence of statistical ensembles for the HMF model (contrary to 3D gravitational systems), the existence of second order phase transitions (instead of first order or zeroth order phase transitions for 3D gravitational systems) and the vanishing of the collision operator at the order $1/N$ in the BBGKY hierarchy contrary to the Coulombian or Newtonian case.

The paper is organized as follows. In Section 2, we consider the statistical equilibrium states of the HMF model in both microcanonical and canonical ensembles. We synthesize previously known results and we derive explicit criteria of thermodynamical stability for the uniform phase as well as for the clustered phase. We also describe corrections to the mean-field approximation close to the critical point. In Section 3, we consider a one-dimensional gaseous system with cosine interactions (the analogue of a "gaseous star") described by the Euler equations with a barotropic equation of state. We discuss in particular the equivalent of the Jeans instability. In Section 4, we consider the collisionless evolution of the HMF model (the analogue of a "stellar system") described by the Vlasov equation and discuss the concept of violent relaxation and metaequilibrium states. We interprete these quasi-equilibrium states as particular stationary solutions of the Vlasov equation on the coarse-grained scale resulting from phase mixing and incomplete violent relaxation. We regard Tsallis functional $S_q[f] = -\frac{1}{q-1}\int(f^q - f)d\theta dv$ and Boltzmann functional $S_B[f] = -\int f\ln f d\theta dv$ as particular H-functions in the sense of Tremaine et al. [14] associated with polytropic and isothermal distributions. We study the dynamical stability of stationary solutions of the Vlasov equation and compare with the dynamical stability of stationary solutions of the barotropic Euler equations. This is the same type of comparison as between "gaseous systems" and "stellar systems" in astrophysics. In that respect, we discuss the equivalent of the Antonov first law [15] for the HMF model. We derive a criterion of nonlinear dynamical stability for steady states of the Vlasov equation of the form $f = f(\epsilon)$ with $f'(\epsilon) < 0$ where ϵ is the individual energy, and show that it can be written as a condition on the velocity of sound in the corresponding barotropic gas. This criterion is equivalent to the criterion obtained by Yamaguchi et al. [16] but it is expressed differently. We also analyze the linear dynamical stability of steady states of the Vlasov equation and study the dispersion relation for isothermal and polytropic distributions. In Section 5, we discuss the collisional evolution of the HMF model and explain why the kinetic theory is more complicated than for 3D Newtonian interactions. In particular, the Landau and the Lenard-Balescu collision terms vanish for $1D$ systems so that the evolution of the system as a whole is due to terms of order smaller than $1/N$ in the expansion of the correlation functions for $N \to +\infty$. This implies that the relaxation time is larger than Nt_D (where t_D is the dynamical time). By contrast, we can develop a kinetic theory at order $1/N$ to analyze the relaxation of a "test particle" in a bath of "field particles" with static distribution $f_0(v)$. The evolution of the distribution function $P(v,t)$ of the velocity of the test particle satisfies a Fokker-Planck equation. We give explicit expressions for the diffusion coefficient and the auto-correlation function and we compare their expressions depending on whether collective effects are taken into account or not. We also show that the auto-correlation function decreases exponentially rapidly in time with a rate coinciding with the damping rate γ of a stable perturbed solution of the Vlasov equation. Finally, in Section 6, we discuss the case of self-attracting Brownian particles described by non-local Fokker-Planck equations. This stochastic model is the canonical counterpart of the Hamiltonian N-body problem. It could be called the BMF (Brownian Mean Field) model. We study the dynamical stability of steady states of the non-local Smoluchowski equation and solve this equation numerically to show the formation of a clustered state from an unstable homogeneous state due to long-range interactions. We also provide analytical solutions of the dynamics close to the critical point T_c and for $T=0$. All these models have an equivalent in the astrophysical literature and this paper stresses the analogies and differences between self-gravitating systems and the HMF model. This comparative study brings new light in the statistical mechanics of self-gravitating systems by showing what is specific to gravity and what is common to systems with long-range interactions.

2 Statistical equilibrium

2.1 The mean-field approximation

We consider a system of N particles moving on a circle and interacting via a cosine binary potential. This is the so-called HMF model. As explained in the Introduction, this model can also describe a system of stars moving on elliptical orbits, each orbit exerting a torque on the others. Fundamentally, the dynamics of this system is governed by the Hamilton equations

$$m_i\frac{d\theta_i}{dt}=\frac{\partial H}{\partial v_i},\qquad m_i\frac{dv_i}{dt}=-\frac{\partial H}{\partial \theta_i},$$
$$H=\sum_{i=1}^{N}\frac{1}{2}m_i v_i^2-\frac{k}{4\pi}\sum_{i\neq j}m_i m_j\cos(\theta_i-\theta_j), \tag{1}$$

where θ_i is the angle that makes particle/ellipse i with an axis of reference and k is the coupling constant (similar to the gravitational constant G). In the rest of the paper, we shall refer to this system as a "stellar system"; this is to emphasize the analogies with real 3D stellar systems whose dynamics is also governed by Hamiltonian equations with long-range interactions. We have also generalized the usual HMF model to a population of particles with different masses m_i. However, in most of the paper, we shall assume that all the particles have the same mass $m=1$. The multi-species HMF model will be discussed specifically in Section 7.

The evolution of the N-body distribution function is governed by the Liouville equation

$$\frac{\partial P_N}{\partial t}+\sum_{i=1}^{N}\left(v_i\frac{\partial P_N}{\partial \theta_i}+F_i\frac{\partial P_N}{\partial v_i}\right)=0 \tag{2}$$

where $F_i=-\frac{k}{2\pi}\sum_{j=1}^{N}\sin(\theta_i-\theta_j)$ is the force experienced by particle i. Any distribution of the form $P_N=\chi(H)\delta(E-H)$ is a stationary solution of the Liouville equation. For $N\gg 1$ (fixed) and $t\to+\infty$, this system is expected to reach a statistical equilibrium state due to the development of correlations between particles (this will be referred to as a "collisional" relaxation). As is customary in statistical mechanics, we shall assume that the equilibrium N-body distribution function is described by the microcanonical distribution

$$P_N(\theta_1,v_1,...,\theta_N,v_N)=\frac{1}{g(E)}\delta(E-H), \tag{3}$$

expressing that all accessible microstates (with the right values of energy and mass) are equiprobable. Whether this is indeed the case has not been proved rigorously as it relies on a hypothesis of ergodicity, so this statement is essentially a *postulate*.

For systems with long-range interactions (self-gravitating systems, 2D vortices, HMF model,...), it can be shown that the mean-field approximation is *exact* in an appropriate thermodynamic limit. This can be shown for example by considering an equilibrium BBGKY-like hierarchy [17,18]. For the HMF model, the thermodynamic limit is $N\to+\infty$ in such a way that the properly normalized energy $\epsilon=8\pi E/kM^2$ and temperature $\eta=\beta kM/4\pi$ are fixed, where $M=Nm$ is the total mass. These control parameters are similar to those, $\epsilon=ER/GM^2$ and $\eta=\beta GMm/R$, describing 3D gravitational systems [19]. In that limit $N\to+\infty$, the two-body distribution function can be expressed as a product of two one-body distribution functions

$$P_2(\theta_1,v_1,\theta_2,v_2)=P_1(\theta_1,v_1)P_1(\theta_2,v_2)+O(1/N). \tag{4}$$

The average density of particles in phase space is given by $f(\theta,v)=\langle\sum_i\delta(\theta-\theta_i)\delta(v-v_i)\rangle=NP_1(\theta,v)$. The total mass can then be expressed as

$$M=\int f d\theta dv. \tag{5}$$

On the other hand, the average energy $E=\langle H\rangle$ is

$$E=N\int P_1(\theta,v)\frac{v^2}{2}d\theta dv$$
$$-\frac{k}{4\pi}N(N-1)\int\cos(\theta-\theta')P_2(\theta,v,\theta',v')d\theta dv d\theta' dv'. \tag{6}$$

In the mean-field limit, it reduces to

$$E=\frac{1}{2}\int f v^2 d\theta dv+\frac{1}{2}\int f\Phi d\theta dv, \tag{7}$$

where

$$\Phi(\theta) = -\frac{k}{2\pi}\int_0^{2\pi} \cos(\theta-\theta')\rho(\theta')d\theta', \quad (8)$$

is the potential and $\rho = \int f dv$ is the spatial density. Note that the average force experienced by a particle located in θ is $\langle F\rangle = -\Phi'(\theta)$. If ρ is symmetric with respect to the x-axis, so that $\rho(-\theta) = \rho(\theta)$, the foregoing relation can be rewritten

$$\Phi(\theta) = B\cos\theta, \quad (9)$$

where

$$B = -\frac{k}{2\pi}\int_0^{2\pi} \rho(\theta')\cos\theta' d\theta'. \quad (10)$$

This parameter B is the equivalent of the magnetization (usually denoted M) in the case of spin systems. Inserting the relation (9) in equation (7), we find that the energy can be rewritten

$$E = \frac{1}{2}\int f v^2 d\theta dv - \frac{\pi B^2}{k}, \quad (11)$$

so that the potential energy is directly expressed in terms of B.

2.2 The Boltzmann entropy

We wish to determine the macroscopic distribution of particles at statistical equilibrium, assuming that all accessible microstates (with given E and M) are equiprobable. To that purpose, we divide the μ-space $\{\theta, v\}$ into a very large number of microcells with size h. We do not put any exclusion, so that a microcell can be occupied by an arbitrary number of particles. We shall now group these microcells into macrocells each of which contains many microcells but remains nevertheless small compared to the phase-space extension of the whole system. We call ν the number of microcells in a macrocell. Consider the configuration $\{n_i\}$ where there are n_1 particles in the 1st macrocell, n_2 in the 2nd macrocell etc. Using the standard combinatorial procedure introduced by Boltzmann, the number of microstates corresponding to the macrostate $\{n_i\}$, i.e. its probability, is given by

$$W(\{n_i\}) = N!\prod_i \frac{\nu^{n_i}}{n_i!}. \quad (12)$$

This is the Maxwell-Boltzmann statistics. As is customary, we define the entropy of the state $\{n_i\}$ by

$$S(\{n_i\}) = \ln W(\{n_i\}). \quad (13)$$

It is convenient here to return to a representation in terms of the distribution function giving the phase-space density in the ith macrocell $f_i = f(\theta_i, v_i) = n_i/\nu h$. Using the Stirling formula and passing to the continuum limit $\nu \to 0$, we obtain the usual expression of the Boltzmann entropy

$$S_B[f] = -\int f\ln f d\theta dv, \quad (14)$$

up to some unimportant additive constant. Then, the statistical equilibrium state, corresponding to the most probable distribution of particles, is obtained by maximizing the Boltzmann entropy (14) at fixed mass M and energy E, i.e.

$$\text{Max}\ \{S_B[f] \quad | \ E[f] = E, M[f] = M\}. \quad (15)$$

This maximization problem defines the microcanonical equilibrium state, which is the correct description for an isolated Hamiltonian system.

We shall also consider the canonical description which applies to a system in contact with a thermostat imposing its temperature T. We will give an example of canonical system in Section 6 corresponding to Brownian particles in interaction described by stochastic (not Hamiltonian) equations (BMF model). In the canonical ensemble, the statistical equilibrium state is obtained by minimizing the free energy $F_B[f] = E[f] - TS_B[f]$ at fixed mass M and temperature T, i.e.

$$\text{Min}\ \{F_B[f] \quad | \ M[f] = M\}. \quad (16)$$

The relation between the Boltzmann entropy $S_B[f]$ and the density of states $g(E)$ in the microcanonical ensemble and between the Boltzmann free energy $F_B[f]$ and the partition function $Z(\beta)$ in the canonical ensemble is discussed in Chavanis [17,18]. The variational problems (15) and (16) correspond to a saddle point approximation in the functional integral representation of $g(E)$ and $Z(\beta)$.

The variational problem (15) has been first investigated, in the HMF context, by Inagaki [8]. We review and precise the main results of his study and present an alternative derivation of the condition of thermodynamical stability using methods similar to those introduced by Padmanabhan [2] and Chavanis [20,19] for 3D self-gravitating systems. This will make the analogy between the two systems (stellar systems and HMF model) closer. This will also allow us to study the thermodynamical stability of the clustered phase, while the analysis of Inagaki [8] is restricted to the uniform phase.

2.3 The first variations: the Maxwell-Boltzmann distribution

We need first to determine the critical points of entropy at fixed mass and energy. We write the variational principle in the form

$$\delta S_B - \beta\delta E - \alpha\delta M = 0, \quad (17)$$

where $\beta = 1/T$ (inverse temperature) and α (chemical potential) are Lagrange multipliers enforcing the constraints on E and M. The solution of (17) is the mean-field Maxwell-Boltzmann distribution

$$f = A' e^{-\beta(\frac{v^2}{2}+\Phi)}, \quad (18)$$

where Φ depends on f through equation (8). The foregoing relation is therefore an integro-differential equation. Integrating over the velocity, we obtain the mean-field Boltzmann distribution

$$\rho = A e^{-\beta\Phi}. \quad (19)$$

The same distributions (18) and (19) are obtained in the canonical ensemble by cancelling the first variations of free energy at fixed mass, using $\delta F_B - \alpha \delta M = 0$. Therefore, the critical points of the variational problems (15) and (16) coincide. In the microcanonical ensemble, we need to relate the Lagrange multiplier β to the energy E. This defines a *series of equilibria* $\beta = \beta(E)$. In the canonical ensemble, the inverse temperature β is assumed given and the corresponding (average) energy is obtained by inverting the graph $\beta(E)$. Of course, only the stable part of the series of equilibria is of physical interest, and defines the *caloric curve* (see Sect. 2.5). We note that the equilibrium distributions (18) and (19) can also be obtained from an equilibrium BBGKY-like hierarchy in the thermodynamic limit $N \to +\infty$ defined in Section 2.1 (Chavanis [17,18]).

Using the relation (9), the distribution of particles at statistical equilibrium is given by

$$\rho = A e^{-\beta B \cos\theta}. \tag{20}$$

The axis of symmetry is determined by the initial conditions. If $B = 0$, the density ρ is uniform. This defines the homogeneous phase. If $B \neq 0$, we have inhomogeneous states with one cluster at $\theta = 0$ (if $B < 0$) or at $\theta = \pi$ (if $B > 0$). The constant A is related to the mass by

$$M = 2\pi A I_0(\beta B), \tag{21}$$

where I_n are the modified Bessel functions

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)d\theta. \tag{22}$$

For $z \to 0$,

$$I_n(z) = (\frac{1}{2}z)^n \left[\frac{1}{\Gamma(n+1)} + \frac{z^2}{4\Gamma(n+2)} + ...\right], \tag{23}$$

and for $z \to +\infty$,

$$I_n(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{4n^2-1}{8z} + ...\right]. \tag{24}$$

Using equations (10) and (22) we find that the order parameter B is determined as a function of the temperature β by the implicit equation

$$B = \frac{kM}{2\pi}\frac{I_1(\beta B)}{I_0(\beta B)}. \tag{25}$$

Setting $x = \beta B$, we can rewrite the foregoing relation in the form

$$\frac{4\pi T}{kM}x = 2\frac{I_1(x)}{I_0(x)}. \tag{26}$$

Then x, and consequently B, is determined as a function of T by a simple graphical construction sketched in Figure 1. We see that $B = 0$ is always solution although $B \neq 0$ is possible only if

$$T < \frac{kM}{4\pi} \equiv T_c. \tag{27}$$

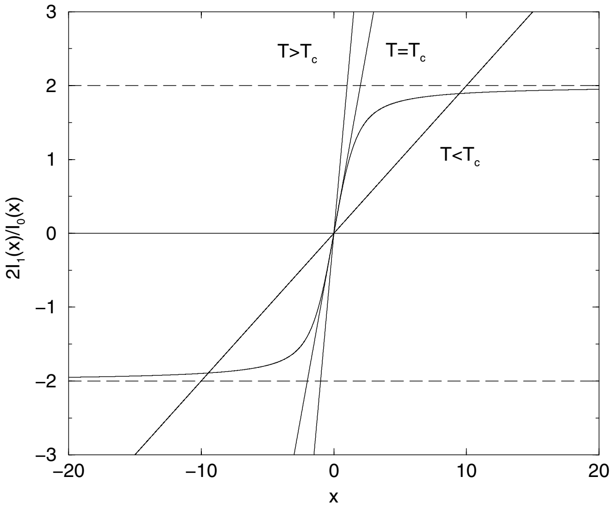

Fig. 1. Graphical construction showing the appearance of a clustered phase below some critical temperature T_c.

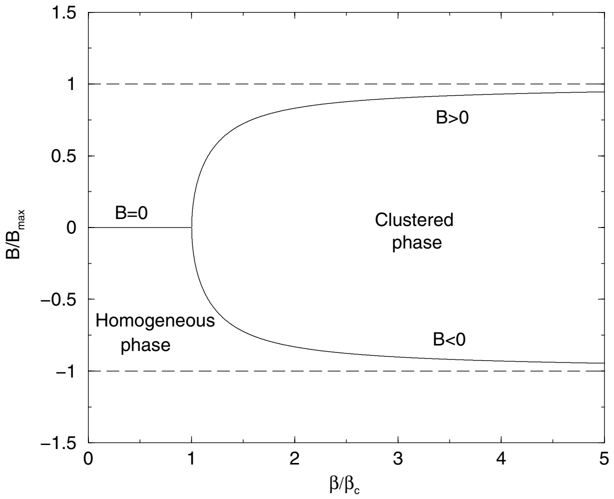

Fig. 2. Order parameter B (magnetization) as a function of the inverse temperature.

In terms of the energy (11) this corresponds to

$$E < \frac{kM^2}{8\pi} \equiv E_c. \tag{28}$$

The function $B(T)$ is shown in Figure 2 and its asymptotic behaviours are given in Section 2.4. Figure 2 displays a second order phase transition. We have a situation similar to a gravitational collapse below a critical temperature T_c or below a critical energy E_c. For $T > T_c$, the system is homogeneous. For $T < T_c$, the system forms one cluster around $\theta = 0$ (for $B < 0$) or around $\theta = \pi$ (for $B > 0$). At $T = 0$, the equilibrium state is a Dirac peak $\rho = M\delta(\theta - \pi)$ (for $B = B_{max}$). Density profiles are plotted in Figure 3 for different values of $x = \beta B(\beta)$. Using the stellar disk interpretation of Pichon Lynden-Bell, we have represented some stellar orbits in Figures 4 and 5 by randomly choosing the orbits' angles with the equilibrium distribution $\rho(\theta)$. The "disk phase" for $T > T_c$ is represented in Figure 4 and the "bar phase" for $T < T_c$ is represented in Figure 5.

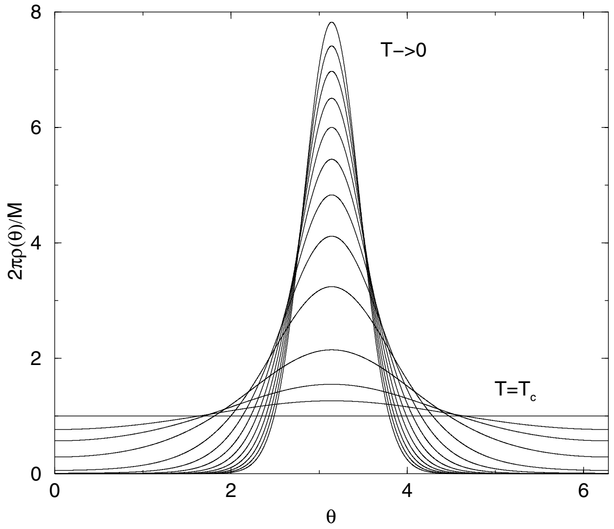

Fig. 3. Evolution of the density profile as temperature is decreased (from bottom to top).

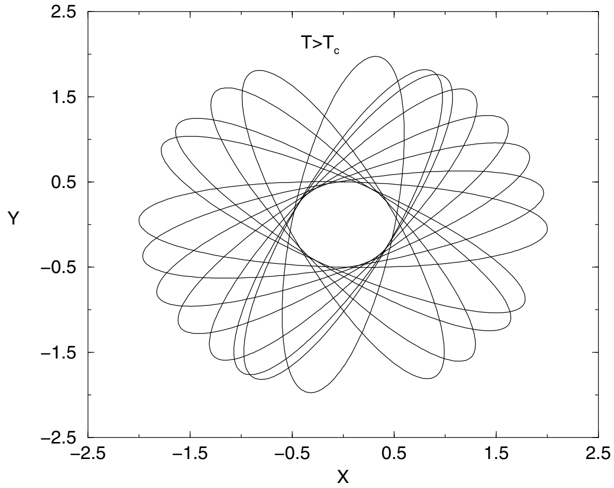

Fig. 4. Stellar orbits in the "disk phase" for $T > T_c$.

2.4 The thermodynamical parameters

According to equation (26), the relation between the temperature and the order parameter can be written in dimensionless form as

$$\eta \equiv \beta/\beta_c = \frac{x}{2}\frac{I_0(x)}{I_1(x)}. \tag{29}$$

For $x \to 0$,

$$\eta = 1 + \frac{x^2}{8} + ... \tag{30}$$

and for $x \to +\infty$,

$$\eta = \frac{x}{2}\left(1 + \frac{1}{2x} + ...\right). \tag{31}$$

Returning to original variables, we deduce that

$$\frac{B}{B_{max}} = \pm\sqrt{2\left(1 - \frac{T}{T_c}\right)}, \qquad \left(0 < \frac{T_c - T}{T_c} \ll 1\right), \tag{32}$$

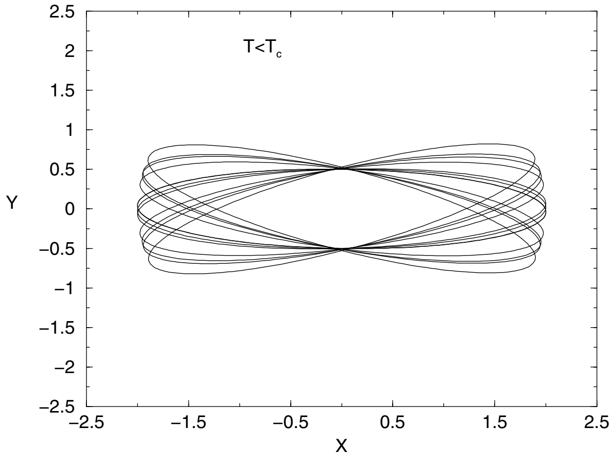

Fig. 5. Stellar orbits in the "bar phase" for $T < T_c$.

$$\frac{B}{B_{max}} = \pm\left(1 - \frac{T}{4T_c}\right), \qquad (T \ll T_c), \tag{33}$$

where $B_{max} = 2T_c = kM/2\pi$ is the maximum value of the magnetization obtained for $T = 0$ when all the particles are at $\theta = \pi$. With this notation, the parameter x can be written

$$x = 2\eta B/B_{max}. \tag{34}$$

On the other hand, for the Maxwellian velocity distribution (18), the expression of the energy (11) becomes

$$E = \frac{1}{2}MT - \frac{\pi B^2}{k}. \tag{35}$$

In terms of dimensionless parameters, we get

$$\epsilon \equiv E/E_c = \frac{1}{\eta}\left(1 - \frac{x^2}{2\eta}\right). \tag{36}$$

For the homogeneous phase $B = 0$, we simply have

$$\epsilon = \frac{1}{\eta}. \tag{37}$$

For the inhomogeneous phase, we can easily obtain asymptotic expansions. For $x \to 0$,

$$\epsilon = 1 - \frac{5}{8}x^2 + ... \tag{38}$$

and for $x \to +\infty$,

$$\epsilon = -2\left(1 - \frac{2}{x} + ...\right). \tag{39}$$

Returning to original variables, we deduce that

$$\frac{E}{E_0} = \frac{1}{2}\frac{T}{T_c}, \qquad (T > T_c), \tag{40}$$

$$\frac{E}{E_0} = \frac{1}{2}\left(6 - 5\frac{T}{T_c}\right), \qquad \left(0 < \frac{T_c - T}{T_c} \ll 1\right), \tag{41}$$

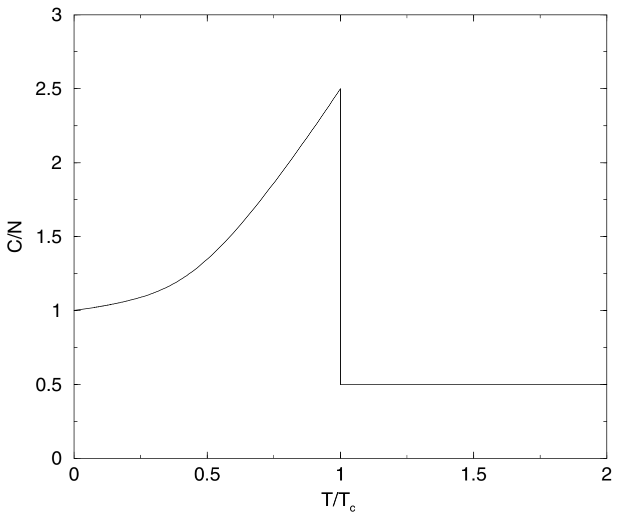

Fig. 6. Specific heat $C = dE/dT$ as a function of temperature. It experiences a discontinuity at the critical temperature T_c.

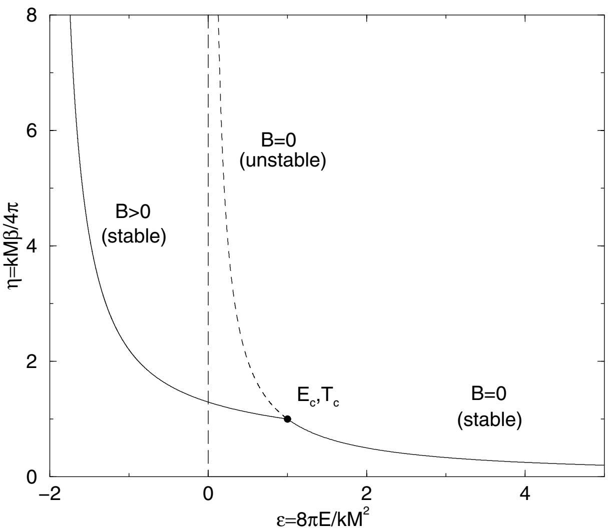

Fig. 7. Calorique curve (series of equilibria) for the HMF model. The system displays a second order phase transition at a critical point E_c, T_c.

$$\frac{E}{E_0} = \frac{T}{T_c} - 1, \qquad (T \ll T_c), \tag{42}$$

where $-E_0 = -kM^2/4\pi$ is the minimum value of energy obtained for $T = 0$ ($\epsilon_0 = -2$). For $T \to T_c^-$, the specific heat $C = dE/dT$ is given by $C = \frac{5}{2}M$ and for $T > T_c$ by $C = \frac{M}{2}$. Therefore, at the critical point, it experiences a discontinuity (see Fig. 6):

$$C(T_c^-) - C(T_c^+) = 2M. \tag{43}$$

The caloric curve/series of equilibria $\beta(E)$ is shown in Figure 7. It displays a second order phase transition at $(\epsilon, \eta) = (1, 1)$. This is different from 3D gravitational systems which rather display first order and zeroth order phase transitions (see, e.g., Chavanis [21]).

2.5 The second variations: thermodynamical stability

To analyze the thermodynamical stability of the solutions determined by the variational problems (15) and (16), we use an approach similar to that followed by Padmanabhan [2] and Chavanis [20,19] in the case of 3D self-gravitating systems. We first maximize $S_B[f]$ at fixed $M[f]$, $E[f]$ and $\rho(\theta)$. This gives the Maxwellian

$$f(\theta, v) = \frac{1}{\sqrt{2\pi T}} \rho(\theta) e^{-\frac{v^2}{2T}}. \tag{44}$$

Then, we can re-express the entropy and the energy as a function of the density in such a way that

$$S_B = \frac{1}{2} M \ln T - \int \rho \ln \rho \, d\theta, \tag{45}$$

$$E = \frac{1}{2} MT + \frac{1}{2} \int \rho \Phi d\theta. \tag{46}$$

We now take the variations of entropy around an equilibrium solution. To second order

$$\delta S_B = \frac{M}{2} \frac{\delta T}{T} - \frac{M}{4} \left(\frac{\delta T}{T} \right)^2 - \int \delta\rho \ln \rho d\theta - \frac{1}{2} \int \frac{(\delta\rho)^2}{\rho} d\theta. \tag{47}$$

Now, the conservation of energy implies

$$0 = \delta E = \frac{1}{2} M \delta T + \int \Phi \delta\rho d\theta + \frac{1}{2} \int \delta\rho \delta\Phi d\theta. \tag{48}$$

Eliminating δT, we find that

$$\delta^2 S_B = -\frac{1}{2T} \int \delta\rho \delta\Phi d\theta - \frac{1}{MT^2} \left(\int \Phi \delta\rho d\theta \right)^2 - \frac{1}{2} \int \frac{(\delta\rho)^2}{\rho} d\theta. \tag{49}$$

We define the quantity q by the relation

$$\delta\rho = \frac{dq}{d\theta}. \tag{50}$$

Physically, $q = \int_0^\theta \delta\rho d\theta$ represents the mass perturbation within the interval $[0, \theta]$. Then, the conservation of mass is equivalent to $q(0) = q(2\pi) = 0$. Inserting this relation in equation (49) and using integrations by parts, we can put the second order variations of entropy in the quadratic form

$$\delta^2 S_B = \int_0^{2\pi} \int_0^{2\pi} d\theta d\theta' q(\theta) K(\theta, \theta') q(\theta'), \tag{51}$$

with

$$K(\theta, \theta') = -\frac{1}{MT^2} \frac{d\Phi}{d\theta}(\theta) \frac{d\Phi}{d\theta}(\theta') + \frac{k}{4\pi T} \sin(\theta - \theta') \frac{d}{d\theta'} + \frac{1}{2} \delta(\theta - \theta') \frac{d}{d\theta'} \left[\frac{1}{\rho(\theta')} \frac{d}{d\theta'} \right]. \tag{52}$$

We are thus led to consider the eigenvalue problem

$$\int_0^{2\pi} K(\theta,\theta')q_\lambda(\theta')d\theta' = \lambda q_\lambda(\theta). \quad (53)$$

This yields

$$\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{dq}{d\theta}\right) + \frac{k}{2\pi T}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = \frac{2V}{MT^2}\frac{d\Phi}{d\theta} + 2\lambda q, \quad (54)$$

where

$$V = \int_0^{2\pi} \frac{d\Phi}{d\theta} q(\theta)d\theta. \quad (55)$$

The system is stable if all $\lambda < 0$ and unstable if at least one $\lambda > 0$. So far, we have worked in the microcanonical ensemble. If we work in the canonical ensemble, we have to minimize the free energy $F_B = E - TS_B$ at fixed mass and temperature. We can easily check that fixing T in the preceding calculations amounts to taking $V = 0$. Thus, instead of equation (54), we obtain

$$\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{dq}{d\theta}\right) + \frac{k}{2\pi T}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = 2\lambda q. \quad (56)$$

2.6 The condition of thermodynamical stability

If we consider the stability of the uniform solution $\rho = M/2\pi$ and $\Phi = 0$, the foregoing equations simplify into

$$\frac{2\pi}{M}\frac{d^2q}{d\theta^2} + \frac{k}{2\pi T}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = 2\lambda q. \quad (57)$$

The eigenvalue equation is the same in the two ensembles. Hence, the stability criteria coincide, implying that the statistical ensembles (microcanonical and canonical) are equivalent. This is at variance with the case of 3D stellar systems [2,20,19].

We can study the solutions of equation (57) by decomposing q in Fourier series. For the mode n, we have $q_n = A_n \sin(n\theta)$. For $n \neq 1$, we get $\lambda_n = -\frac{\pi n^2}{M} < 0$ showing that these modes do not induce instability. For $n = 1$, we have $\lambda_1 = \frac{k}{4T} - \frac{\pi}{M}$. The uniform solution will be unstable if $\lambda_1 > 0$ yielding condition (27). Therefore, the uniform phase is stable for $T > T_c$ while it is unstable for $T < T_c$. By using the theory of linear series of equilibria (Katz [22–24]), applied here to a bifurcation point, we directly conclude from the inspection of Figure 7 that the clustered phase will be stable for $T < T_c$ when the homogeneous phase becomes unstable.

More precisely, it is possible to solve equations (54) and (56) analytically for the clustered phase in the limit $B \to 0$, which is valid close to the critical point (E_c, T_c). The calculations are detailed in Appendix A. In the canonical ensemble ($V = 0$), it is found that the largest eigenvalue is

$$\lambda M = -2\pi\left(\frac{\beta}{\beta_c} - 1\right), \quad (58)$$

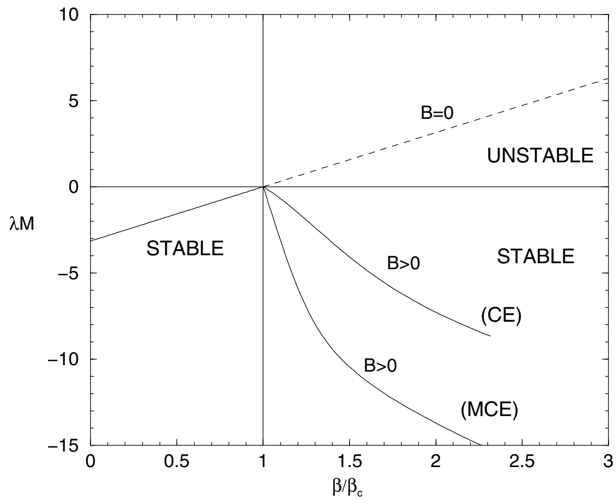

Fig. 8. Dependence of the largest eigenvalue λ with the temperature. A negative value of λ corresponds to stability ($\delta^2 S < 0$) and a positive value of λ corresponds to instability.

and in the microcanonical ensemble ($V \neq 0$) that

$$\lambda M = -10\pi\left(\frac{\beta}{\beta_c} - 1\right). \quad (59)$$

More generally, the exact value of λ obtained by solving equations (54) and (56) numerically is plotted versus the inverse temperature in Figure 8. Since $\lambda < 0$, we check explicitly that the clustered phase is stable.

2.7 Correction to the mean-field approximation close to the critical point

We can obtain the expression of the two-points correlation function from an equilibrium BBGKY-like hierarchy by closing the second equation of the hierarchy with the Kirkwood approximation [17,18]. This is valid to order $1/N$ in the thermodynamic limit defined previously. For the HMF model, it is then possible to obtain an explicit expression of the correlation function in the homogeneous phase. Writing the two-body distribution function as

$$N^2 P_2(\theta_1 - \theta_2) = \rho^2[1 + h(\theta_1 - \theta_2)], \quad (60)$$

it is found that [17]:

$$h(\theta_1 - \theta_2) = \frac{2}{N}\frac{\beta/\beta_c}{1 - \beta/\beta_c}\cos(\theta_1 - \theta_2). \quad (61)$$

We note that the correlation function diverges close to the critical point $\beta \to \beta_c$ where the clustered phase appears and the homogeneous phase becomes unstable. This implies that the mean-field approximation ceases to be valid close to the critical point. We expect a similar result for 3D self-gravitating systems although the situation is more difficult to analyze as (real) self-gravitating systems are always inhomogeneous.

If we take into account the contribution of non-trivial pair correlations (61) in the potential energy, we find furthermore that equation (37) is replaced by

$$\epsilon = \frac{1}{\eta} - \frac{1}{N}\frac{2\eta}{1-\eta}. \qquad (62)$$

Therefore, finite N effects modify the shape of the caloric curve in the vicinity of the critical point. The mean-field approximation is valid if $N(1-\eta) \gg 1$. When the mean field approximation is valid, its order one correction for the specific heat is

$$C = \frac{N}{2}\left[1 + \frac{1}{\pi N}\frac{1}{(T/T_c - 1)^2}\right] \qquad (T > T_c). \qquad (63)$$

Finally, it is found that the spatial correlations of the force are given by [17,18]:

$$\langle F(0)F(\theta)\rangle = \frac{\rho k^2}{4\pi}\frac{1}{1-\beta/\beta_c}\cos\theta. \qquad (64)$$

In particular, the variance of the force is

$$\langle F^2\rangle = \frac{\rho k^2}{4\pi}\frac{1}{1-\beta/\beta_c}. \qquad (65)$$

Note that without the correlations, we would have simply obtained $\langle F^2\rangle = \frac{\rho k^2}{4\pi}$ which corresponds to the high temperature limit ($T \to +\infty$) of equation (65).

For the HMF model, the variance (65) of the force is finite while the variance of the Newtonian force for 3D self-gravitating systems is infinite (Chandrasekhar and von Neumann [25]). For the HMF model, the distribution of the force is normal (Gaussian) while the distribution of the gravitational force in $D = 3$ is a particular Lévy law called the Holtzmark distribution. On the other hand, for 2D point vortices, the variance of the velocity is a marginal Gaussian distribution intermediate between normal and Lévy laws (Chavanis and Sire [26]). Therefore, these three systems with long-range interactions (self-gravitating systems, 2D vortices and HMF model) have their own specificities despite their overall analogies.

3 Gaseous systems

As indicated in the Introduction, the HMF model is similar to stellar systems in astrophysics. In Sections 4 and 5, we shall discuss the kinetic theory of the HMF model and obtain the equivalent of the Vlasov and Landau equations that are used to describe the dynamics of elliptical galaxies and globular clusters respectively. However, in order to facilitate the discussion and the comparison, it is useful to discuss first the dynamics of a one-dimensional barotropic fluid system with cosine interactions described by the Euler equations. In astrophysics, these equations describe the dynamics of barotropic stars. Stellar systems and barotropic stars are often treated in parallel due to their analogies [15]. In particular, it is possible to infer sufficient conditions of dynamical stability for spherical stellar systems from the dynamical stability of a barotropic star with the same density distribution. This constitutes the Antonov first law. Therefore, it is also of interest to develop this parallel in the case of the HMF model. To have a similar vocabulary, the systems considered in this paper will also be called "stellar systems" and "gaseous stars" although they are only one-dimensional and correspond to a cosine interaction.

3.1 Euler equations and energy functional

We consider a gaseous system described by the Euler equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\theta}(\rho u) = 0, \qquad (66)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial\theta} = -\frac{1}{\rho}\frac{\partial p}{\partial\theta} - \frac{\partial\Phi}{\partial\theta}, \qquad (67)$$

where the potential Φ is given by (8). To close the equations, we consider an arbitrary barotropic equation of state $p = p(\rho)$. We emphasize that these equations cannot be derived from the HMF model (1) which rather leads to kinetic equations like the Vlasov equation. However, we shall see that there is a close connection between the stationary states of the Vlasov and the Euler equations and that the limits of dynamical stability are the same in the two systems. Thus, the study of the Euler equation (which is simpler than the Vlasov equation) brings many information about the stability of stationary states of the HMF model with respect to the Vlasov equation even if the Euler system does not describe dynamically the HMF model.

It is straightforward to verify that the energy functional

$$\mathcal{W} = \int\rho\int_0^\rho \frac{p(\rho')}{\rho'^2}d\rho' d\theta + \frac{1}{2}\int\rho\Phi d\theta + \int\rho\frac{u^2}{2}d\theta, \qquad (68)$$

is conserved by the Euler equations ($\dot{\mathcal{W}} = 0$). The first term is the internal energy, the second the potential energy and the third the kinetic energy associated with the mean motion. The mass is also conserved. Therefore, a minimum of $\mathcal{W}$ at fixed mass determines a stationary solution of the Euler equations which is formally nonlinearly dynamically stable in the sense of Holm et al. [27]. We are led therefore to consider the minimization problem

$$\text{Min}\ \{\mathcal{W}[\rho, u] \quad | \ M[\rho] = M\}. \qquad (69)$$

3.2 First variations: the condition of hydrostatic equilibrium

Cancelling the first order variations of equation (68), we obtain $u = 0$ and the condition of hydrostatic equilibrium

$$\frac{dp}{d\theta} = -\rho\frac{d\Phi}{d\theta}. \qquad (70)$$

Therefore, extrema of $\mathcal{W}$ correspond to stationary solutions of the Euler equations (66–67). On the other hand, combining the condition of hydrostatic equilibrium (70) and the equation of state $p = p(\rho)$, we get

$$\int^{\rho} \frac{p'(\rho')}{\rho'} d\rho' = -\Phi, \tag{71}$$

so that ρ is a function of Φ that we note $\rho = \rho(\Phi)$. Using equation (9), we find that

$$\rho = \rho(B\cos\theta), \tag{72}$$

where B is determined by the implicit equation

$$B = -\frac{k}{2\pi}\int_0^{2\pi} \rho(B\cos\theta')\cos\theta' \; d\theta'. \tag{73}$$

Again, $B = 0$ is a solution of this equation characterizing a homogeneous phase $\rho = M/2\pi$. To determine the point of bifurcation to the inhomogeneous phase, we expand equation (73) around $B = 0$. Then, we find that cluster solutions appear when

$$1 + \frac{k}{2}\frac{d\rho}{d\Phi}(0) \leq 0. \tag{74}$$

Using the condition of hydrostatic balance (70), this can be rewritten

$$c_s^2 \leq (c_s^2)_{crit} = \frac{kM}{4\pi}, \tag{75}$$

where $c_s = (dp/d\rho)^{1/2}$ is the velocity of sound in the homogeneous phase where $\rho = M/2\pi$.

3.3 Second variations: the condition of nonlinear dynamical stability

The second variation of $\mathcal{W}$ due to variation of the velocity is trivially positive. The second variation of $\mathcal{W}$ due to variation of ρ is

$$\delta^2\mathcal{W} = \frac{1}{2}\int \delta\rho\delta\Phi d\theta + \int \frac{p'(\rho)}{2\rho}(\delta\rho)^2 d\theta, \tag{76}$$

which must be positive for stability. Using the same procedure as in Section 2.5, we find that the eigenvalue equation determining the stability of the solution is now

$$\frac{d}{d\theta}\left(\frac{p'(\rho)}{\rho}\frac{dq}{d\theta}\right) + \frac{k}{2\pi}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = 2\lambda q, \tag{77}$$

and that the condition of stability is $\lambda < 0$ (this yields a maximum of $-\mathcal{W}$). For the uniform solution $\rho = M/2\pi$, we can repeat exactly the same steps as in Section 2.6 since $p'(\rho)$ is a constant c_s^2 which plays the role of T in the thermodynamical analysis. Therefore, we find that the uniform phase is formally nonlinearly dynamically stable with respect to the Euler equations when

$$c_s^2 \geq \frac{kM}{4\pi}, \tag{78}$$

and dynamically unstable otherwise. According to equation (75), the onset of dynamical instability coincides with the point where the clustered phase appears.

The stability of the clustered phase can be investigated by solving the eigenvalue equation (77) for a specified equation of state $p(\rho)$. This equation is the counterpart of equation (222) of Chavanis [19] for 3D self-gravitating gaseous spheres.

3.4 The condition of linear stability: Jeans-like criterion

We now study the linear dynamical stability of a stationary solution of the Euler equation. We consider a small perturbation around a stationary solution of equations (66–67) and write $\rho = \rho + \delta\rho$, $u = \delta u$ etc... The linearized equations for the perturbation are

$$\frac{\partial\delta\rho}{\partial t} + \frac{\partial}{\partial\theta}(\rho\delta u) = 0, \tag{79}$$

$$\rho\frac{\partial\delta u}{\partial t} = -\frac{\partial}{\partial\theta}(p'(\rho)\delta\rho) - \rho\frac{\partial\delta\Phi}{\partial\theta} - \delta\rho\frac{\partial\Phi}{\partial\theta}, \tag{80}$$

$$\delta\Phi(\theta) = -\frac{k}{2\pi}\int_0^{2\pi}\cos(\theta-\theta')\delta\rho(\theta')d\theta'. \tag{81}$$

Writing the time dependence in the form $\delta\rho \sim e^{\lambda t}$,..., we get

$$\lambda\delta\rho + \frac{d}{d\theta}(\rho\delta u) = 0, \tag{82}$$

$$\lambda\rho\delta u = -\frac{d}{d\theta}(p'(\rho)\delta\rho) - \rho\frac{d\delta\Phi}{d\theta} - \delta\rho\frac{d\Phi}{d\theta}. \tag{83}$$

Introducing the notation (50), the continuity equation can be integrated into

$$\lambda q + \rho\delta u = 0, \tag{84}$$

where we have imposed $\delta u(0) = \delta u(2\pi) = 0$. Substituting this relation in equation (83) and using the condition of hydrostatic equilibrium (70), we finally obtain

$$\frac{d}{d\theta}\left(\frac{p'(\rho)}{\rho}\frac{dq}{d\theta}\right) + \frac{k}{2\pi}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = \frac{\lambda^2}{\rho}q. \tag{85}$$

This equation is the counterpart of the Eddington equation of pulsations for a barotropic star (see also Eq. (224) of Chavanis [19]). We note that equations (77) and (85) coincide for the neutral point $\lambda = 0$. Therefore, the conditions of linear stability and formal nonlinear dynamical stability coincide. The same conclusion holds for 3D barotropic stars [19].

Considering the uniform phase $\rho = M/2\pi$ and following a method similar to that developed in Section 2.6, we find that the most destabilizing mode ($n = 1$) is

$$\delta\rho = a_1 \cos\theta\, e^{\lambda t}, \quad \delta u = -\frac{2\pi\lambda}{M} a_1 \sin\theta\, e^{\lambda t}, \tag{86}$$

where the growth rate is given by

$$\lambda^2 = \frac{kM}{4\pi} - c_s^2. \tag{87}$$

When $c_s^2 \leq kM/4\pi$, then $\lambda = \pm\sqrt{\lambda^2}$ and the perturbation grows exponentially rapidly (unstable case). When $c_s^2 \geq kM/4\pi$, then $\lambda = \pm i\sqrt{-\lambda^2}$ and the perturbation oscillates with a pulsation $\omega = \sqrt{-\lambda^2}$ without attenuation (stable case). Therefore, the uniform phase is linearly (and also formally nonlinearly) dynamically stable with respect to the Euler equations when

$$c_s^2 \geq \frac{kM}{4\pi}, \tag{88}$$

and linearly dynamically unstable otherwise.

3.5 Particular examples

3.5.1 Isothermal gas

For an isothermal gas, we have

$$p = \rho T, \qquad c_s^2 = T, \tag{89}$$

and

$$\mathcal{W} = T\int \rho \ln\rho d\theta + \frac{1}{2}\int \rho\Phi d\theta + \int \rho\frac{u^2}{2} d\theta, \tag{90}$$

where the temperature T is uniform. We note that the energy functional (90) of an isothermal gas coincides with the Boltzmann free energy $F_B[\rho] = E[\rho] - TS_B[\rho]$ of a N-body system in the canonical ensemble, see equations (45) and (46) of Section 2.5. This remark also holds for 3D self-gravitating systems [20]. The pulsation equation (85) becomes

$$\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{dq}{d\theta}\right) + \frac{k}{2\pi T}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = \frac{\lambda^2}{T\rho}q, \tag{91}$$

which can be connected to equation (56). According to the criteria (78–88), the uniform phase is formally nonlinearly dynamically stable for

$$T \geq T_c \equiv \frac{kM}{4\pi}, \tag{92}$$

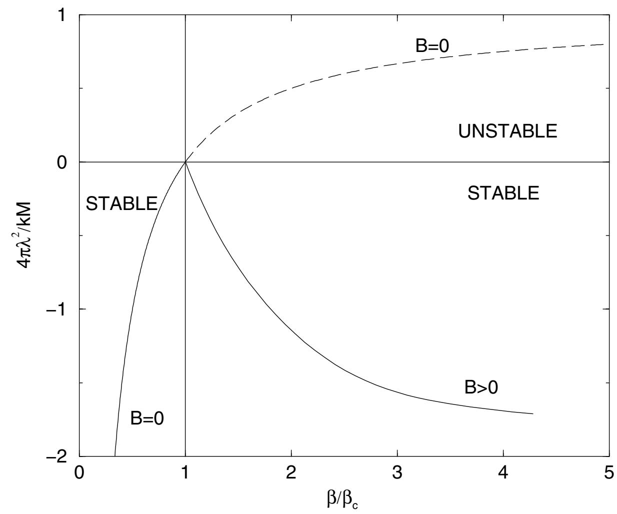

Fig. 9. Growth rate and pulsation period of an isothermal gas as a function of the temperature.

and linearly dynamically unstable otherwise. This criterion (or more generally the criterion (88)) can be regarded as the counterpart of the Jeans instability criterion in astrophysics [15]. We emphasize, however, an important difference. In the case of 3D self-gravitating systems, the Jeans criterion selects a critical wavelength λ_J (increasing with the temperature) above which the system is unstable against gravitational collapse. In the present context, where the interaction is truncated to one Fourier mode $n = 1$, the criterion (92) selects a critical temperature below which the system is unstable. The generalization of the Jeans instability criterion for an arbitrary binary potential of interaction in D dimensions is discussed in Appendix C and in Chavanis [17]. This generalization clearly shows the connection between the HMF model and 3D self-gravitating systems.

According to equation (87), the relation between λ and the temperature T is

$$\lambda^2 = T_c - T, \qquad T_c = \frac{kM}{4\pi}. \tag{93}$$

For $T < T_c$, the growth rate is $\lambda = (T_c - T)^{1/2}$ and for $T > T_c$, the pulsation is $\omega = (T - T_c)^{1/2}$. Following the preceding remark, we stress that, in the present context, λ and ω only depend on the temperature T, while for a 3D gravitational gas, they depend on the wavelength of the perturbation [15]. Here, the unstable mode is fixed to $n = 1$.

Considering now the clustered phase and using a perturbative approach similar to that of Appendix A for $T \to T_c^-$ (not detailed), we find that the pulsation is given by $\omega = \sqrt{2(T_c - T)}$. For smaller temperatures, the eigenvalue equation can be solved numerically and the results are shown in Figure 9.

3.5.2 Polytropic gas

For a polytropic gas, we have

$$p = K\rho^{\gamma}, \qquad \gamma = 1 + \frac{1}{n}, \tag{94}$$

where K is the polytropic constant and n is the polytropic index. For $n \to +\infty$, we recover the isothermal case with $\gamma = 1$ and $K = T$. For that reason K is sometimes called a polytropic temperature. The energy functional (68) can be written

$$\mathcal{W} = \frac{K}{\gamma - 1}\int(\rho^{\gamma} - \rho)d\theta + \frac{1}{2}\int \rho\Phi d\theta + \int \rho\frac{u^2}{2}d\theta. \tag{95}$$

We have added a constant term (proportional to the total mass) in the polytropic energy functional (95) so as to recover the isothermal energy functional (90) for $n \to +\infty$. Under this form, we note that the energy functional of a polytropic gas has the same form as the Tsallis free energy $F_{\gamma}[\rho] = E[\rho] - KS_{\gamma}[\rho]$ where γ plays the role of the q-parameter and K the role of a generalized temperature. The same remark holds for 3D self-gravitating systems. However, this resemblance is essentially fortuitous and the mark of a *thermodynamical analogy* [19,28].

If we define the local temperature by $p(\theta) = \rho(\theta)T(\theta)$, we obtain $T(\theta) = K\rho(\theta)^{1/n}$ and $c_s^2(\theta) = \gamma T(\theta)$. We note that, for a polytropic distribution, the kinetic temperature $T(\theta)$ usually depends on the position while the polytropic temperature K is uniform as in an isothermal gas. However, in the uniform phase $T = K\rho^{1/n}$ is a constant that can be called the temperature of the polytropic gas. The velocity of sound in the homogeneous phase is

$$c_s^2 = \gamma T = K\gamma\rho^{\gamma-1} = K\frac{1+n}{n}\left(\frac{M}{2\pi}\right)^{1/n}. \tag{96}$$

The condition of dynamical stability (78–88) can be written

$$K \geq K_n \equiv \frac{kM}{4\pi}\frac{n}{1+n}\left(\frac{2\pi}{M}\right)^{1/n}, \tag{97}$$

or, equivalently,

$$T \geq T_n \equiv \frac{T_c}{\gamma} = \frac{kM}{4\pi\gamma}. \tag{98}$$

For $\gamma > 1$ (i.e., $n > 0$), the critical temperature T_n is smaller than the corresponding one for an isothermal gas $T_c = T_{\infty}$, i.e. the instability is delayed. For $\gamma < 1$ (i.e., $n < 0$), the instability is advanced. Similar results are obtained for 3D gravitational systems [28]. According to equation (87), the relation between λ and the kinetic temperature T is

$$\lambda^2 = \gamma(T_n - T). \tag{99}$$

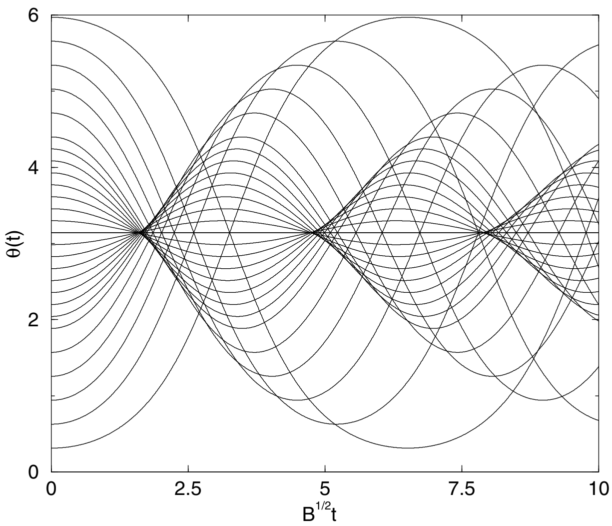

Fig. 10. Characteristics of the forced Burgers equation (102) showing the appearance of shocks and chevrons.

3.6 The local Euler equation

We can consider a simplified problem where the potential in equation (67) is fixed to its equilibrium value $\Phi = B\cos\theta$. In that case, we get the local Euler equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} + B\sin\theta. \tag{100}$$

The stationary solutions are given by (72). The linear stability of a stationary solution amounts to solving the Sturm-Liouville problem

$$\frac{d}{d\theta}\left(\frac{p'(\rho)}{\rho}\frac{dq}{d\theta}\right) = \frac{\lambda^2}{\rho}q. \tag{101}$$

This problem will be considered in Section 6.3 for the isothermal equation of state.

In the pressureless case, we get the forced Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \theta} = B\sin\theta. \tag{102}$$

It can be solved by the method of characteristics, writing the equation of motion of a particle as

$$\frac{d^2\theta}{dt^2} = B\sin\theta. \tag{103}$$

The trajectories $\theta(t)$ can then be expressed in terms of elliptic functions. The dynamics of the forced Burgers equation is interesting as it develops "shocks" and "chevrons" (caustics) singularities (see Fig. 10). The Burgers equation also appears in cosmology to describe the formation of large-scale structures in the universe (Vergassola et al. [29]). A detailed description of this dynamics is given by Barré et al. [30] in the context of the repulsive HMF model. In that case, the forced Burgers equation (102) with $\sin(2\theta)$ instead of $\sin\theta$ models the short time dynamics of the Hamiltonian N-body system. For the attractive HMF model (1), the short time dynamics can be modelled by the *non-local* Euler equation (67) with zero pressure $p = 0$. This is solution of the Vlasov equation (see Sect. 4) with $f(\theta, v, t) = \rho(\theta, t)\delta(v - u(\theta, t))$. This single-speed solution is valid until the first shock. However, the connection with the *local* Euler equation (102) is not clear in that case because the homogeneous phase is unstable which precludes the possibility of deriving (102) from (67) as is done in Barré et al. [30] in the repulsive case. Note also that in the attractive HMF model (ferromagnetic) we just have one cluser while the repulsive HMF model (anti-ferromagnetic) shows a bicluster.

4 Violent relaxation, metaequilibrium states and dynamical stability of collisionless stellar systems

We now come back to the HMF model defined by the Hamilton equations (1) and develop a kinetic theory by analogy with stellar systems. In particular, we emphasize the importance of the Vlasov equation and the concept of violent relaxation introduced by Lynden-Bell [11].

4.1 Vlasov equation and H-functions

For systems with long-range interactions, the relaxation time toward the statistical equilibrium state (18) is larger than Nt_D, where t_D is the dynamical time [17]. Accordingly, for $N \to +\infty$, the relaxation time is extremely long and, for timescales of physical interest, the evolution of the system is essentially *collisionless*. More precisely, for $t \ll t_{relax}$ and $N \to +\infty$ (Vlasov limit), the time dependence of the distribution function is governed by the Vlasov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \theta} - \frac{\partial \Phi}{\partial \theta}\frac{\partial f}{\partial v} = 0, \qquad (104)$$

which has to be solved in conjunction with equation (8). This system of equations is similar to the Vlasov-Poisson system describing the dynamics of elliptical galaxies and other collisionless stellar systems in astrophysics. Starting from an unstable initial condition, the HMF model (1) will achieve a *metaequilibrium* state (on a coarse-grained scale) as a result of phase mixing and violent relaxation [15]. This metaequilibrium state is a particular stationary solution of the Vlasov equation. Since it results from a complex mixing, it is highly robust and nonlinearly dynamically stable with respect to collisionless perturbations. The process of violent relaxation and the convergence of the distribution function toward a stationary solution of the Vlasov equation has been illustrated numerically by Yamaguchi et al. [16] for the HMF model. This is similar to the violent relaxation of stellar systems in astrophysics and 2D vortices in hydrodynamics [3].

One question of great importance is whether we can *predict* the metaequilibrium state achieved by the system as a result of violent relaxation. Lynden-Bell [11] has tried to make such a prediction by resorting to a new type of statistical mechanics accounting for the conservation of all the Casimirs imposed by the Vlasov equation. This theory was developed for the gravitational interaction, but the general ideas and formalism apply to any system with long-range interactions described by the Vlasov equation. In the non-degenerate limit, he predicts a Boltzmann distribution of the form $\overline{f} \sim e^{-\beta\epsilon}$ where the individual mass of the particles does not appear. Lynden-Bell [11] also understood that his statistical prediction is limited by the concept of *incomplete relaxation*. The system tries to reach the most mixed state but, as the fluctuations become weaker and weaker as we approach equilibrium, it can settle on a stable stationary solution of the Vlasov equation which is not the most mixed state. In order to quantify the importance of mixing, Tremaine et al. [14] have introduced the concept of H-functions

$$S = -\int C(\overline{f})d\theta dv, \qquad (105)$$

where C is an arbitrary convex function. The H-functions calculated with the coarse-grained distribution function $\overline{f}$ increase as a result of phase mixing in the sense that $S[\overline{f}(\theta, v, t)] \geq S[\overline{f}(\theta, v, 0)]$ for $t > 0$ where it is assumed that, initially, the system is not mixed so that $\overline{f}(\theta, v, 0) = f(\theta, v, 0)$. This is similar to the H-theorem in kinetic theory. However, contrary to the Boltzmann equation, the Vlasov equation does not single out a unique functional (the above inequality is true for all H-functions) and the time evolution of the H-functions is not necessarily monotonic (nothing is implied concerning the relative values of $H(t)$ and $H(t')$ for $t, t' > 0$). On the other hand, any stationary solution of the Vlasov equation of the form $f = f(\epsilon)$ with $f'(\epsilon) < 0$ extremizes a H-function at fixed mass and energy. If, in addition, it *maximizes* S at fixed E, M, then it is nonlinearly dynamically stable with respect to the Vlasov equation. In astrophysics, such distribution functions depending only on the energy describe spherical stellar systems. This is a particular case of the Jeans theorem [15]. For a 1D system such as the HMF model, this is the general form of inhomogeneous stationary solutions of the Vlasov equation. Therefore, we expect that the H-functions will increase during violent relaxation until one of them (non-universal) reaches its maximum value at fixed mass and energy when a stationary solution of the Vlasov equation is reached (this is not necessarily the case in astrophysics since the system can reach a steady state that does not depend only on energy). Note that the Boltzmann and the Tsallis functionals are particular H-functions (not thermodynamical entropies in that context) associated with particular stationary solutions of the Vlasov equation: isothermal stellar systems and stellar polytropes [28]. All these ideas, first developed for stellar systems, apply to other systems with long-range interactions such as the HMF model.

4.2 Nonlinear dynamical stability criterion for the Vlasov equation

The theory of violent relaxation explains how a collisionless system out of mechanical equilibrium can reach a steady solution of the Vlasov equation on a very short timescale due to long-range interactions and chaotic mixing. Since this metaequilibrium state is stable with respect to collisionless perturbations, it is of interest to determine a criterion of formal nonlinear dynamical stability for the Vlasov equation. For the HMF model, this has been considered by Yamaguchi et al. [16] using the Casimir-Energy method. We shall propose another derivation of the stability criterion which uses a formal analogy with the thermodynamical analysis developed in Section 2 and which is also applicable to the clustered phase. This approach is similar to the one developed by Chavanis [19] for 3D stellar systems.

Let us introduce the functional $S = -\int C(f)d\theta dv$ where $C(f)$ is a convex function. This functional is a particular Casimir so it is conserved by the Vlasov equation. The energy E and the mass M are also conserved. Therefore, a maximum of S at fixed mass M and energy E determines a stationary solution $f(\theta, v)$ of the Vlasov equation that is nonlinearly dynamically stable. We are led therefore to consider the maximization problem

$$\mathrm{Max}\ \{S[f] \quad | \ E[f] = E, M[f] = M\}. \qquad (106)$$

We also note that $F[f] = E[f] - TS[f]$ (where T is a positive constant) is conserved by the Vlasov equation. Therefore, a minimum of F at fixed mass M is nonlinearly dynamically stable with respect to the Vlasov equation (F is called an energy-Casimir functional). This corresponds to the formal stability criterion of Holm et al. [27]. This criterion can be written

$$\mathrm{Min}\ \{F[f] \quad | \ M[f] = M\}. \qquad (107)$$

To study the nonlinear dynamical stability of collisionless stellar systems, we are thus led to consider the optimization problems (106) and (107). These are similar to the conditions of thermodynamical stability (15) and (16) but they involve a more general functional $S[f]$ than the Boltzmann entropy. In addition, they have a completely different interpretation since they determine the nonlinear dynamical stability of a steady solution of the Vlasov equation, not the thermodynamical stability of the statistical equilibrium state. Due to this formal resemblance, we can develop a *thermodynamical analogy* [31] and use an effective thermodynamical vocabulary to investigate the nonlinear dynamical stability of a collisionless stellar system. In this analogy, S plays the role of an effective entropy, T plays the role of an effective temperature and F plays the role of an effective free energy. The criterion (106) is similar to a condition of microcanonical stability and the criterion (107) is similar to a condition of canonical stability.

We also note that the stability criterion (106) is consistent with the phenomenology of violent relaxation. Indeed, the H-functions increase on the coarse-grained scale while the mass and the energy are approximately conserved. Therefore, the metaequilibrium state is expected to maximize a certain H-function (non-universal) at fixed mass and energy. In that interpretation, f has to be viewed as the coarse-grained distribution function $\overline{f}$, not the distribution function itself. The point is that during mixing $D\overline{f}/Dt \neq 0$ and the H-functions $S[\overline{f}]$ increase. Once it has mixed $D\overline{f}/Dt = 0$ so that $\dot{S}[\overline{f}] = 0$. Since $\overline{f}(\theta, v, t)$ has been brought to a maximum $\overline{f}_0(\theta, v)$ of a certain H-function and since $S[\overline{f}]$ is conserved (after mixing), then $\overline{f}_0$ is a nonlinearly dynamically stable steady state of the Vlasov equation.

4.3 First variations: stationary solutions of the Vlasov equation

Introducing Lagrange multipliers as in Section 2.3, the critical points of the variational problems (106) and (107) are given by

$$C'(f) = -\beta\epsilon - \alpha, \qquad (108)$$

where $\epsilon = \frac{v^2}{2} + \Phi$ is the energy of a particle. Since C' is a monotonically increasing function of f, we can inverse this relation to obtain

$$f = F(\beta\epsilon + \alpha), \qquad (109)$$

where $F(x) = (C')^{-1}(-x)$. We can check that any DF $f = f(\epsilon)$ is a stationary solution of the Vlasov equation (104). From the identity

$$f'(\epsilon) = -\beta/C''(f), \qquad (110)$$

resulting from equation (108), we see that $f(\epsilon)$ is a monotonic function of the energy. Assuming that $f(\epsilon)$ is decreasing, which is the physical situation, imposes $\beta = 1/T > 0$.

We note also that for each stellar system with $f = f(\epsilon)$, there exists a corresponding barotropic star with the same equilibrium density distribution. Indeed, defining the density and the pressure by $\rho = \int_{-\infty}^{+\infty} f dv = \rho(\Phi)$, $p = \int_{-\infty}^{+\infty} f v^2 dv = p(\Phi)$, and eliminating the potential Φ between these two expressions, we find that $p = p(\rho)$. Writing explicitly the density and the pressure in the form

$$\rho = 2\int_{\Phi}^{+\infty} F(\beta\epsilon + \alpha)\frac{1}{[2(\epsilon - \Phi)]^{1/2}} d\epsilon, \qquad (111)$$

$$p = 2\int_{\Phi}^{+\infty} F(\beta\epsilon + \alpha)[2(\epsilon - \Phi)]^{1/2} d\epsilon, \qquad (112)$$

and taking the θ-derivative of equation (112), we obtain the condition of hydrostatic equilibrium (70).

Due to the analogy between stellar systems and barotropic stars, it becomes possible to use the results obtained in Section 3 to study the stationary solutions of the Vlasov equation (104). In particular, the transition from homogeneous ($B = 0$) to inhomogeneous ($B \neq 0$)

solutions is again given by the criterion (75). Now, in the case of stellar systems, it is more relevant to express this criterion in terms of the distribution function. Using the identity (299), the criterion (75) determining the appearance of the clustered phase is equivalent to

$$1+\frac{k}{2}\int_{-\infty}^{+\infty}\frac{f'(v)}{v}dv\le 0. \tag{113}$$

We will soon see how this quantity is related to the dielectric function of a gravitational plasma.

4.4 Second variations: the condition of nonlinear dynamical stability

We shall investigate the formal nonlinear dynamical stability of stationary solutions of the Vlasov equation by using the criterion (107). This criterion is less refined than the criterion (106) because all solutions of (107) are solution of (106), but the reciprocal is wrong in general. In particular, for long-range interactions, the optimization problems (106) and (107) may not coincide. In thermodynamics, this corresponds to a situation of ensemble inequivalence [32]. Therefore, the criterion (107) can only give a *sufficient* condition of nonlinear dynamical stability for stationary solutions of the Vlasov equation of the form $f=f(\epsilon)$ with $f'(\epsilon)<0$. This corresponds to the criterion of formal nonlinear dynamical stability given by Holm et al. [27]. The more refined criterion (106) has been introduced by Ellis et al. [33] in 2D hydrodynamics (for the 2D Euler-Poisson system) and applied by Chavanis [19] in stellar dynamics (for the Vlasov-Poisson system).

To obtain a manageable criterion of dynamical stability, we use the same procedure as the one developed in Chavanis [19]. We shall not repeat the steps that are identical. We first minimize the functional $F[f]$ at fixed temperature *and* density $\rho(\theta)$. This gives an optimal distribution $f_*(\theta,v)$, determined by $C'(f_*)=-\beta\frac{v^2}{2}-\lambda(\theta)$, which depends on the density $\rho(\theta)$ through the Lagrange multiplier $\lambda(\theta)$. Then, after some manipulations [19], we can show that the functional $F[\rho]=F[f_*]$ can be put in the form

$$F=\frac{1}{2}\int\rho\Phi d\theta+\int\rho\int_0^{\rho}\frac{p(\rho')}{\rho'^2}d\rho' d\theta, \tag{114}$$

where $p(\rho)$ is the equation of state determined by $C(f)$ according to equations (111) and (112). We now need to minimize $F[\rho]$ at fixed mass. To that purpose, we just have to observe that $F[\rho]$ corresponds to the energy functional (68) of a barotropic gas with $u=0$. Therefore, the cancellation of the first variations of equation (114) returns the condition of hydrostatic equilibrium (70) and the positivity of the second variations leads to the stability criterion $\lambda<0$ linked to the eigenvalue equation (77). Therefore, the criterion of formal nonlinear dynamical stability (107) for the Vlasov equation (stellar systems) is equivalent to the criterion of formal nonlinear dynamical stability (69) for the Euler equations (gaseous barotropic stars).

Using the results of Section 3, we conclude that the uniform phase is formally nonlinearly dynamically stable with respect to the Vlasov equation when

$$c_s^2\ge\frac{kM}{4\pi}. \tag{115}$$

When the inequality is reversed, the uniform phase is a saddle point of F at fixed mass M and we shall see that it is linearly dynamically unstable. Using the identity (299), the nonlinear criterion (115) can be rewritten as

$$1+\frac{k}{2}\int_{-\infty}^{+\infty}\frac{f'(v)}{v}dv\ge 0, \tag{116}$$

which was found by Yamaguchi et al. [16] using a different method. An advantage of the present approach is that this approach is also applicable to an inhomogeneous system. Indeed, the stability of the clustered phase can be investigated by solving the eigenvalue equation (77) for the equation of state specified by the function $C(f)$, and investigating the sign of λ.

4.5 About the Antonov first law

As discussed previously, the criterion (107) providing a condition of nonlinear dynamical stability for a stellar system with $f=f(\epsilon)$ and $f'(\epsilon)<0$ is equivalent to the criterion (69) determining the nonlinear dynamical stability of a barotropic star with the same equilibrium density distribution. On the other hand, we have already indicated that the criterion (107) is less refined than the criterion (106) which is believed to be the strongest criterion of nonlinear dynamical stability for stationary solutions of the Vlasov equation of the form $f=f(\epsilon)$ with $f'(\epsilon)<0$. In general, the criterion (107) just provides a sufficient condition of nonlinear dynamical stability. Thus, we can "miss" stable solutions if we use just (107) instead of (106) [said differently, the set of solutions of (107) is included in (106)]. From these remarks, we conclude that "a stellar system is stable whenever the corresponding barotropic gas is stable" but the converse is wrong in general. This is the so-called Antonov first law in astrophysics [15]. Our approach provides an *extension* of the Antonov first law to the case of nonlinear dynamical stability (while the usual Antonov first law corresponds to linear dynamical stability). Furthermore, by developing a *thermodynamical analogy*, we have provided an original interpretation of the nonlinear Antonov first law in terms of "ensembles inequivalence" for systems with long-range interactions [19].

For 3D self-gravitating systems, we know that the ensembles are *not* equivalent so that the criterion (107) is more restrictive than the criterion (106). In that case, (107) is just a *sufficient* condition of nonlinear dynamical stability (definitely). For example, using the criterion (69), it can be shown that polytropic stars with index $n<3$ are nonlinearly dynamically stable with respect to the Euler equations while polytropic stars with index $3<n<5$ are dynamically unstable (polytropes with index $n>5$

have infinite mass). Therefore, using the "canonical" criterion (107), we can only deduce that stellar polytropes with index $n < 3$ are nonlinearly dynamically stable with respect to the Vlasov equation. However, using the "microcanonical" criterion (106), we can prove that all stellar polytropes with index $n < 5$ are nonlinearly dynamically stable with respect to the Vlasov equation. Polytropes with index $3 < n < 5$ lie in a region of "ensemble inequivalence" (in the thermodynamical analogy) where the "specific heat" is negative [19].

For the HMF model considered in this paper, we believe that the criteria (106) and (107) determine the same set of solutions so that the ensembles are equivalent in that case (no solution is forgotten by (107)). We have shown in Section 2 that this is at least the case for isothermal distributions. If we take for granted that this equivalence extends to any functional (105), we conclude that, for the HMF model, "a stellar system is stable if, and only if, the corresponding barotropic gas is stable". This would be the HMF version of the nonlinear Antonov first law. In that case, the stability limits obtained for the Euler equation in Section 3 can be directly applied to the Vlasov equation (they coincide). Clearly, it would be of great interest to derive general criteria telling when the ensembles are equivalent or inequivalent, depending on the form of the functional (105) and on the form of the potential of interaction $u(\mathbf{r}-\mathbf{r}')$.

4.6 The condition of linear stability

We now study the linear dynamical stability of a spatially homogeneous stationary solution of the Vlasov equation described by $f = f(v)$. This problem was first investigated by Inagaki and Konishi [7] and Pichon [10] and more recently by Choi and Choi [34]. We shall complete here their study. Writing the perturbation in the form $\delta f \sim e^{i(n\theta-\omega t)}$ and using standard methods of plasma physics, we obtain the dispersion relation

$$\epsilon(n,\omega) \equiv 1 + \frac{k}{2}(\delta_{n,1} - \delta_{n,-1}) \int \frac{f'(v)}{nv-\omega} dv = 0, \tag{117}$$

where $\epsilon(n,\omega)$ is the dielectric function and the integral must be performed by using the Landau contour. For the destabilizing mode $n = 1$ ($n = -1$ gives the same result), equation (117) reduces to

$$1 + \frac{k}{2} \int \frac{f'(v)}{v-\omega} dv = 0. \tag{118}$$

The condition of marginal stability is $\omega_i = 0$ where ω_i is the imaginary part of $\omega = \omega_r + i\omega_i$. In that case, the integral in equation (118) can be written as

$$1 + \frac{k}{2}\mathcal{P} \int \frac{f'(v)}{v-\omega_r} dv + i\pi\frac{k}{2} f'(\omega_r) = 0, \tag{119}$$

where $\mathcal{P}$ denotes the principal value. Identifying real and imaginary parts, if follows that

$$1 + \frac{k}{2}\mathcal{P} \int \frac{f'(v)}{v-\omega_r} dv = 0,$$
$$f'(\omega_r) = 0. \tag{120}$$

The second relation fixes the frequency of the perturbation and the first equation determines the point of marginal stability in the series of equilibria. The system is linearly dynamically stable if

$$1 + \frac{k}{2}\mathcal{P} \int \frac{f'(v)}{v-\omega_r} dv \geq 0, \tag{121}$$

and linearly dynamically unstable otherwise. Note that $f(v)$ does not need to be symmetrical. However, if $f(v)$ extremizes a H-function at fixed mass and energy, then it has a single maximum at $v = 0$. Therefore, $\omega_r = 0$ according to equation (120) and the criterion of linear dynamical stability (121) coincides with the criterion of formal nonlinear dynamical stability (116).

4.7 Particular examples

We shall now present explicit results for particular stationary solutions of the Vlasov equation. We consider the case of isothermal stellar systems and stellar polytropes.

4.7.1 Isothermal stellar systems

We consider the H-function

$$S = -\int f \ln f d\theta dv, \tag{122}$$

which is similar to the Boltzmann entropy (14) in thermodynamics. However, as explained in Section 4.1, its physical interpretation is different. Its maximization at fixed mass and energy determines a formally nonlinearly dynamically stable stationary solution of the Vlasov equation corresponding to the isothermal distribution function

$$f = Ae^{-\beta\epsilon}. \tag{123}$$

This distribution function has the same form (but a different interpretation) as the statistical equilibrium state (18) of the N-body system.

The barotropic gas corresponding to the isothermal distribution function (123) is the isothermal gas with an equation of state $p = \rho T$ where $T = 1/\beta$. Therefore, the velocity dispersion β^{-1} of an isothermal stellar system is equal to the velocity of sound $c_s^2 = T$ in the corresponding isothermal gas. The functional (114) takes the form

$$F[\rho] = \frac{1}{2}\int \rho\Phi d\theta + T\int \rho \ln \rho d\theta, \tag{124}$$

and the density is related to the potential according to the formula

$$\rho = A'e^{-\beta\Phi}, \tag{125}$$

which can be obtained by extremizing $F[\rho]$ at fixed mass. We can also express the distribution function in terms of the density according to

$$f = \left(\frac{\beta}{2\pi}\right)^{1/2} \rho(\theta) e^{-\beta \frac{v^2}{2}}. \tag{126}$$

According to what has been said in Section 4.5 about the correspondence between stellar systems and barotropic stars, we conclude that the uniform phase of an isothermal stellar system (123) is formally nonlinearly dynamically stable with respect to the Vlasov equation if $T > T_c$ and linearly dynamically unstable otherwise. In terms of the dimensionless parameters $\eta = \frac{kM}{4\pi T}$ and $\epsilon = \frac{8\pi E}{kM^2}$, the conditions of dynamical stability can be written

$$\eta \leq 1, \qquad \epsilon \geq 1. \tag{127}$$

They coincide with the conditions of thermodynamical stability (see Sect. 2).

For the Maxwellian distribution function (126) with uniform density, the dielectric function can be written

$$\epsilon(1,\omega) = 1 - \eta W(\sqrt{\beta}\omega), \tag{128}$$

where

$$W(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{x}{x-z} e^{-\frac{x^2}{2}} dx, \tag{129}$$

is the W-function of plasma physics [35]. This is an analytic function in the upper plane of the complex z plane which is continued analytically into the lower half plane. Explicitly,

$$W(z) = 1 - z e^{-\frac{z^2}{2}} \int_0^z dy e^{\frac{y^2}{2}} + i\sqrt{\frac{\pi}{2}} z e^{-\frac{z^2}{2}}. \tag{130}$$

We look for solutions of the dispersion relation $\epsilon(1,\omega) = 0$ in the form $\omega = i\lambda$ where λ is real. First, we note that

$$\epsilon(1, i\lambda) = 1 - \eta / G\left(\sqrt{\frac{\beta}{2}}\lambda\right), \tag{131}$$

where we have defined the G-function

$$G(x) = \frac{1}{1 - \sqrt{\pi} x e^{x^2} \mathrm{erfc}(x)}. \tag{132}$$

For $x \to 0$, $G(x) = 1 + \sqrt{\pi}x + ...$ For $x \to +\infty$, $G(x) = 2x^2(1 + \frac{3}{2x^2} + ...)$. For $x \to -\infty$, $G(x) \sim -\frac{1}{2\sqrt{\pi}x} e^{-x^2}$. Therefore, the relation between λ and T is given by

$$\eta = G\left(\sqrt{\frac{\beta}{2}}\lambda\right). \tag{133}$$

The case of neutral stability $\omega = 0$ corresponds to $T = T_c$ (or $\eta = 1$). The case of instability ($\lambda > 0$) corresponds to $T < T_c$. The perturbation grows exponentially rapidly as $\delta f \sim e^{\lambda t}$. The growth rate λ is given by equation (133) which can be explicitly written

$$1 - \frac{T_c}{T}\left\{1 - \sqrt{\frac{\pi}{2T}} \lambda e^{\frac{\lambda^2}{2T}} \mathrm{erfc}\left(\frac{\lambda}{\sqrt{2T}}\right)\right\} = 0. \tag{134}$$

For $T \to T_c^-$, we have

$$\lambda \sim \sqrt{\frac{8}{kM}}(T_c - T), \tag{135}$$

and for $T \to 0$, we have

$$\lambda \to \sqrt{\frac{kM}{4\pi}}\left(1 - 3\frac{T}{T_c}\right)^{1/2}. \tag{136}$$

The first term in equation (136) can be deduced directly from equation (118) by using the distribution function $f(v) = \rho\,\delta(v)$ valid at $T = 0$ and integrating by parts. The case of stability ($\lambda < 0$) corresponds to $T > T_c$. The perturbation is damped exponentially rapidly as $\delta f \sim e^{-\gamma t}$ where $\gamma = -\lambda$. This is similar to the Landau damping in plasma physics, except that here there is no pulsation ($\omega_r = 0$). By contrast, in plasma physics, the pulsation ω_r is much larger than the damping rate γ. The damping rate $\gamma = -\lambda$ is given by

$$\eta = F\left(\sqrt{\frac{\beta}{2}}\gamma\right) \tag{137}$$

where we have defined the F-function

$$F(x) = \frac{1}{1 + \sqrt{\pi} x e^{x^2} \mathrm{erfc}(-x)}, \tag{138}$$

such that $F(x) = G(-x)$. For $x \to 0$, $F(x) = 1 - \sqrt{\pi}x + ...$ For $x \to -\infty$, $F(x) \sim 2x^2(1 + \frac{3}{2x^2} + ...)$. For $x \to +\infty$, $F(x) \sim \frac{1}{2\sqrt{\pi}x} e^{-x^2}$. Explicitly,

$$1 - \frac{T_c}{T}\left\{1 + \sqrt{\frac{\pi}{2T}} \gamma e^{\frac{\gamma^2}{2T}} \mathrm{erfc}\left(-\frac{\gamma}{\sqrt{2T}}\right)\right\} = 0. \tag{139}$$

For $T \to T_c^+$, we have

$$\gamma \sim \sqrt{\frac{8}{kM}}(T - T_c), \tag{140}$$

and for $T \to +\infty$, we have

$$\gamma \sim \sqrt{2T \ln T}. \tag{141}$$

Obviously, the relation (134) between the growth rate and the temperature of an isothermal stellar system is different from the corresponding relation (93) valid for an isothermal gas (they coincide only at $T = 0$). A similar distinction is noted in the case of 3D self-gravitating systems. In particular, Figure 11 can be compared with Figure 5.1 of Binney and Tremaine [15]. Moreover, in the unstable regime, a gaseous medium supports sound waves with pulsation $\omega = (T - T_c)^{1/2}$ that are not attenuated.

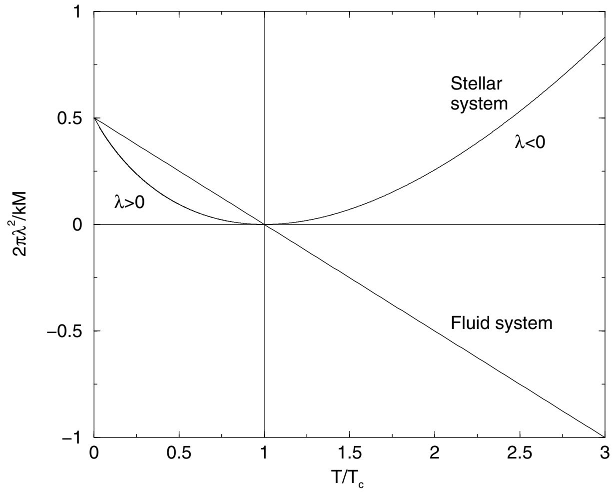

Fig. 11. Growth rate and decay rate of isothermal stellar systems and isothermal stars as a function of the temperature in the framework of the HMF model.

By contrast, in a stellar medium at $T < T_c$, there exists solutions with no wave ($\omega_r = 0$) for which the perturbation is damped exponentially. Other solutions with $\omega_r \neq 0$ probably exist but they are more difficult to investigate analytically. This is left for a future study.

The relations (134) and (139) have been obtained previously by Choi and Choi [34] using a slightly different approach. Our derivation emphasizes the close link with results in plasma physics. In addition, our formalism will be used in Section 5.4 to show that the damping rate γ of a perturbation is equal to the exponential decay of the time auto-correlation function of the force. The generalization of these results for an arbitrary form of long-range potential is given in Chavanis [17].

4.7.2 Stellar polytropes

We consider the H-function

$$S_q = -\frac{1}{q-1}\int (f^q - f)d\theta dv, \tag{142}$$

where q is a real number. This functional has been introduced by Tsallis [36] in non-extensive thermodynamics. The aim was to develop a generalized thermodynamical formalism to describe quasi-equilibrium structures in complex media that are not described by the Boltzmann distribution. In this sense, $S_q[f]$ is interpreted as a generalized entropy and its maximization as a condition of (generalized) thermodynamical stability. In the context of Vlasov systems, we shall rather interprete $S_q[f]$ as a particular H-function (see Sect. 4.1) and its maximization as a condition of nonlinear dynamical stability [1]. Its maximization at fixed mass and energy leads to a particular class of nonlinearly dynamically stable stationary solutions of the Vlasov equation called stellar polytropes. The fact that the criteria (106) and (107) of nonlinear dynamical stability are similar to criteria of generalized thermodynamical stability is the mark of a thermodynamical analogy [31,28].

Stellar polytropes are described by the distribution function

$$f = \left[\mu - \frac{(q-1)\beta}{q}\epsilon\right]^{\frac{1}{q-1}}, \tag{143}$$

obtained from equation (108). When the term in brackets is negative, the distribution function is set equal to $f = 0$. The index n of the polytrope in one dimension is related

[1] If we were to apply Tsallis thermodynamics in the context of violent relaxation, we would need to introduce the fine-grained distribution of phase levels $\rho(\theta, v, \eta)$ [which is the relevant *probability field* in that context] and replace the Lynden-Bell entropy $S_{L.B.}[\rho] = -\int \rho \ln \rho d\theta dv d\eta$ (Lynden-Bell [11]) by $S_q[\rho] = -\frac{1}{q-1}\int (\rho^q - \rho)d\theta dv d\eta$ as suggested in Brands et al. [37]. In that case, $S_q[\rho]$ can be regarded as a generalized entropy trying to take into account non-ergodicity and lack of complete mixing in collisionless systems with long-range interactions. In that point of view, the parameter q measures the efficiency of mixing ($q = 1$ if the system mixes well which is implicitly assumed in Lynden-Bell's statistical theory). In the two-levels approximation $f \in \{0, 1\}$ and in the dilute limit $\overline{f} \ll 1$, $S_q[\rho]$ can be expressed as a functional of the coarse-grained distribution function $\overline{f} \equiv \int \rho \eta d\eta = \rho$ of the form $S_q[\overline{f}] = -\frac{1}{q-1}\int (\overline{f}^q - \overline{f})d\theta dv$. In this particular limit, $S_q[\overline{f}]$ can be interpreted as a thermodynamical entropy generalizing $S[\overline{f}] = -\int \overline{f} \ln \overline{f} d\theta dv$ which is a particular case of the Lynden-Bell entropy for two levels in the dilute limit [38]. In conclusion, Tsallis functional $S_q[\rho]$ expressed in terms of $\rho(\theta, v, \eta)$ is an entropy but Tsallis functional $S_q[\overline{f}]$ expressed in terms of $\overline{f}(\theta, v)$ is either a H-function (dynamics) or the reduced form of entropy $S_q[\rho]$ (thermodynamics) for two levels in the dilute limit. In any case, it is not clear why non-ergodic effects could be encapsulated in the simple functional $S_q[\rho]$ introduced by Tsallis. Tsallis entropy is "natural" because it has mathematical properties very close to those possessed by the Boltzmann entropy and it is probably relevant to describe a certain type of mixing and non-ergodic behaviour as in the case of porous media and weak chaos (it may be seen as an entropy on a fractal phase space). However, many other types of non-ergodic behaviour can occur and other functionals $S = -\int C(\rho)d\theta dv d\eta$ could be relevant. Observation of stellar systems, 2D vortices and quasi-equilibrium states of the HMF model resulting from incomplete violent relaxation do not favour Tsallis distributions in a universal manner. Other distributions can emerge. In fact, we must give up the hope to *predict* the metaequilibrium state in case of incomplete relaxation. We must rather try to construct stable stationary solutions of the Vlasov equation in order to *reproduce* observed phenomena. The H-functions (105) can be useful in that context. An alternative procedure is to keep the Lynden-Bell form of entropy unchanged and develop a *dynamical* theory of violent relaxation [39,40] in order to take into account incomplete mixing through a variable diffusion coefficient related to the strength of the fluctuations. In that case, non-ergodicity is explained by the decay of the fluctuations of Φ driving the relaxation, not by a complex structure of phase-space. Generalized entropies are not necessary in that case.

to the parameter q by the relation [41]:

$$n = \frac{1}{2} + \frac{1}{q-1}. \tag{144}$$

For $n \to +\infty$ (or $q \to 1$), we recover the isothermal distribution function (123) and the H-function (122). Therefore, Tsallis functional connects continuously isothermal and polytropic distributions. Physical polytropic distribution functions (see Chavanis and Sire [41]) have $\beta > 0$ and $q \geq 1$ (i.e. $n \geq 1/2$) or $1/3 < q \leq 1$ (i.e. $n < -1$).

The barotropic gas corresponding to the polytropic distribution function (143) is the polytropic gas

$$p = K\rho^{\gamma}, \qquad \gamma = 1 + \frac{1}{n}, \tag{145}$$

with the polytropic constant

$$K = \frac{1}{n+1}\left\{\sqrt{2}A\frac{\Gamma(1/2)\Gamma(n+1/2)}{\Gamma(n+1)}\right\}^{-1/n}, \; n > \frac{1}{2} \tag{146}$$

$$K = -\frac{1}{n+1}\left\{\sqrt{2}A\frac{\Gamma(1/2)\Gamma(-n)}{\Gamma(1/2-n)}\right\}^{-1/n}, \; n < -1 \tag{147}$$

where $A = (|q-1|\beta/q)^{1/(q-1)}$. In the present context, the polytropic constant K is related to the Lagrange multiplier β. Therefore, K and $T_0 = \beta^{-1}$ play the role of effective temperatures (see Chavanis and Sire [28] for a more detailed discussion). For a polytropic distribution, the functional (114) takes the form

$$F[\rho] = \frac{1}{2}\int \rho\Phi d\theta + \frac{K}{\gamma-1}\int(\rho^{\gamma}-\rho)d\theta, \tag{148}$$

and the relation between the density and the potential is

$$\rho = \left[\lambda - \frac{\gamma-1}{K\gamma}\Phi\right]^{\frac{1}{\gamma-1}}. \tag{149}$$

Comparing equations (143) and (149), we note that a polytropic distribution with index q in phase space yields a polytropic distribution with index $\gamma = 1+2(q-1)/(q+1)$ in physical space. In this sense, polytropic laws are stable laws since they keep the same structure as we pass from phase space $f = f(\epsilon)$ to physical space $\rho = \rho(\Phi)$ as noticed in Chavanis [42]. This is probably the only distribution enjoying this property. Similarly, comparing (142)-(7) and (148) the “free energy” in phase space $F[f] = E[f] - T_0 S_q[f]$ (where $T_0 = 1/\beta$) becomes $F[\rho] = E[\rho] - KS_{\gamma}[\rho]$ in physical space. Morphologically, the polytropic temperature K plays the same role in physical space as the temperature $T_0 = 1/\beta$ in phase space.

We can express the distribution function in terms of the density according to

$$f = \frac{1}{Z}\left[\rho(\theta)^{1/n} - \frac{v^2/2}{(n+1)K}\right]^{n-1/2}, \tag{150}$$

with

$$Z = \sqrt{2}\frac{\Gamma(1/2)\Gamma(n+1/2)}{\Gamma(n+1)}[K(n+1)]^{1/2}, \quad n > \frac{1}{2} \tag{151}$$

$$Z = \sqrt{2}\frac{\Gamma(1/2)\Gamma(-n)}{\Gamma(1/2-n)}[-K(n+1)]^{1/2}, \quad n < -1. \tag{152}$$

Introducing the kinetic temperature (velocity dispersion) $T(\theta) = \langle v^2\rangle = p(\theta)/\rho(\theta) = K\rho(\theta)^{1/n}$, this can be rewritten

$$f = B_n\frac{\rho(\theta)}{\sqrt{2\pi T(\theta)}}\left[1 - \frac{v^2/2}{(n+1)T(\theta)}\right]^{n-1/2}, \tag{153}$$

with

$$B_n = \frac{\Gamma(n+1)}{\Gamma(n+1/2)(n+1)^{1/2}}, \qquad n > \frac{1}{2} \tag{154}$$

$$B_n = \frac{\Gamma(1/2-n)}{\Gamma(-n)[-(n+1)]^{1/2}}, \qquad n < -1. \tag{155}$$

Equation (153) is the counterpart of equation (126) for isothermal systems. For $n > 1/2$, the distribution $f = 0$ for $|v| > v_{max} = \sqrt{2(n+1)T}$.

According to what has been said in Section 4.4 about the correspondence between stellar systems and barotropic stars, we conclude that the uniform phase of a polytropic stellar system (143) with index n is formally nonlinearly dynamically stable with respect to the Vlasov equation if $K \geq K_n$ or $T \geq T_n$ and linearly dynamically unstable otherwise. It can be useful to introduce the dimensionless parameter $\eta = kM/4\pi T$ where $T = K\rho^{1/n}$ is the kinetic temperature in the homogeneous phase where $\rho = M/2\pi$. For $n \to +\infty$ (isothermal case), we recover the dimensionless parameter $\eta = kM/4\pi T$ of Section 4.7.1. On the other hand, in the homogeneous phase ($B = 0$), the energy (7) is given by $E = \frac{1}{2}\int p d\theta = \frac{1}{2}MT$. Therefore, the normalized energy $\epsilon = 8\pi E/kM^2$ is expressed in term of η according to $\epsilon = \frac{1}{\eta}$. In terms of these dimensionless parameters, the uniform phase is formally nonlinearly dynamically stable for

$$\eta \leq \eta_{crit} = \gamma, \qquad \epsilon \geq \epsilon_{crit} = \frac{1}{\gamma}, \tag{156}$$

and linearly dynamically unstable otherwise. Note how the critical energy and temperature are simply expressed in terms of the polytropic index γ. For $n \to +\infty$, we recover the case of isothermal stellar systems with $\epsilon_{crit} = 1$ and $\eta_{crit} = 1$. Note that the line $(\epsilon_{crit}, \eta_{crit})$ coincides with the line $B = 0$ in Figure 7. We thus clearly see how the series of equilibria for polytropic distributions places itself in the (ϵ, η) plane (we just have to displace the critical point $(\epsilon_{crit}, \eta_{crit})$ along the line $B = 0$ as sketched in Fig. 12).

We also emphasize that, using the criterion (115) we have obtained the condition of nonlinear dynamical instability (156) for stellar polytropes with almost no calculation. Of course, the same result can be obtained from the

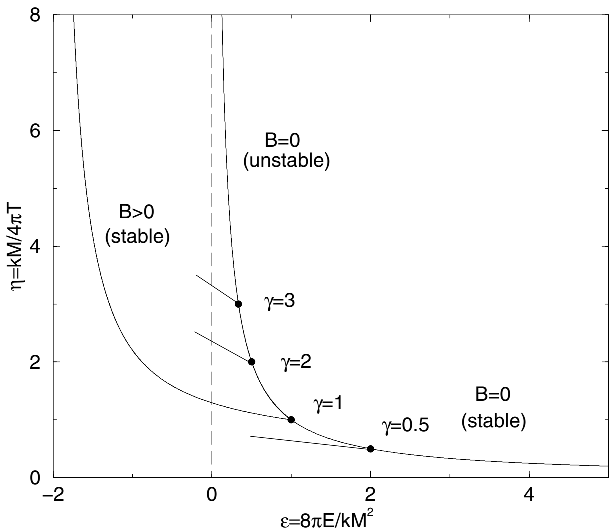

Fig. 12. Bifurcation diagram of stellar polytropes that are stationary solutions of the Vlasov equation. The homogeneous phase is nonlinearly dynamically stable for $\epsilon \geq \epsilon_{crit} = 1/\gamma$ where $\gamma = 1 + 1/n$ and $n = \frac{1}{2} + \frac{1}{q-1}$. It becomes linearly dynamically unstable for $\epsilon < \epsilon_{crit}$ where the branch of clustered states (represented schematically by a line) appears. Isothermal stellar systems correspond to ($q = 1$, $n = \infty$, $\gamma = 1$). In ordinate, T is defined by $K(M/2\pi)^{1/n}$. In the homogeneous phase, it represents the kinetic temperature of a polytropic stellar system. Note that η is a monotonic function of the Lagrange multiplier β so that the curve can be viewed as a series of equilibria of polytropic distributions.

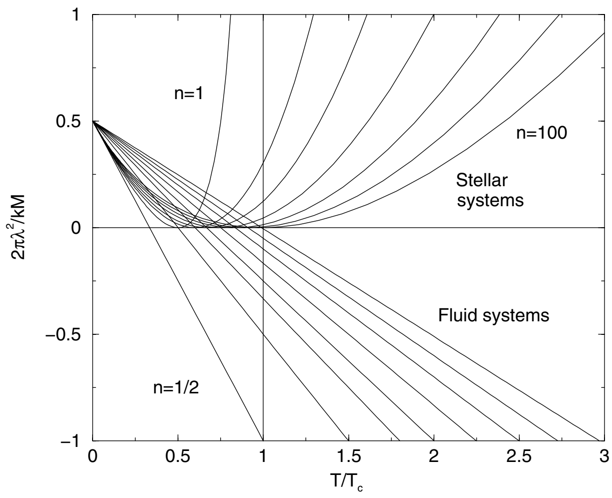

Fig. 13. Growth rate and decay rate for stellar polytropes and polytropic stars in the framework of the HMF model. The index goes from $n = 1$ to $n = 100$. We have also shown the case $n = 1/2$ (water-bag distribution). The critical temperature is smaller than for an isothermal gas ($T_n < T_c$).

criterion (116) by explicitly performing the integral. The criterion of nonlinear dynamical stability (115) that we have found is simpler, albeit equivalent. Moreover, it has a more physical interpretation since it is expressed as a condition on the velocity of sound in a gas with the same equation of state as the original kinetic system.

For the polytropic distribution (153) with uniform density, the dielectric function can be written

$$\epsilon(1,\omega) = 1 - \frac{T_n}{T} W_n(\omega/\sqrt{T}), \tag{157}$$

where we have introduced the function

$$W_n(z) = \frac{1}{\sqrt{2\pi}} \frac{B_n}{n} (n - 1/2) \int \frac{x[1 - \frac{x^2/2}{n+1}]^{n-3/2}}{x - z} dx \tag{158}$$

with $W_n(0) = 1$. The range of integration is such that the term in brackets remains positive. For $n \to +\infty$, we recover the W-function (129). For $\omega = 0$, we obtain the critical temperature $T = T_n$ as in a polytropic gas. As in Section 4.7.1, we look for solutions of the dispersion relation $\epsilon(1,\omega) = 0$ in the form $\omega = i\lambda$ where λ is real. For $T < T_n$, the system is unstable and the growth rate $\lambda > 0$ is given by

$$\frac{T_n}{T} = G_n\left(\frac{\lambda}{\sqrt{2T}}\right), \tag{159}$$

where

$$G_n(x) = \left\{ \frac{B_n}{n\sqrt{\pi}} \left(n - \frac{1}{2}\right) \int \frac{t^2}{t^2 + x^2} \left[1 - \frac{t^2}{n+1}\right]^{n-\frac{3}{2}} dt \right\}^{-1}.$$

For $T > T_n$, the system is stable and the damping rate $\gamma = -\lambda > 0$ is given by

$$\frac{T_n}{T} = F_n\left(\frac{\gamma}{\sqrt{2T}}\right), \tag{160}$$

where

$$F_n(x) = \frac{1}{G_n(x)^{-1} + R_n(x)} \tag{161}$$

with

$$R_n(x) = \frac{2\sqrt{\pi}B_n}{n} \left(n - \frac{1}{2}\right) x \left[1 + \frac{x^2}{n+1}\right]^{n-\frac{3}{2}}. \tag{162}$$

This additional term comes from the residue theorem when the pulsation $\omega = -i\gamma$ lies in the lower half of the complex plane. For $n \to +\infty$, we recover the G and F functions (132) and (138). The dependence of the growth rate and decay rate with the temperature is shown in Figures 13 and 14. For $n = \infty$, we recover the isothermal case of Figure 11.

4.7.3 Fermi or water-bag distribution

For $n = 1/2$, the distribution function (143) is a step function: $f(\epsilon) = \eta_0$ if $-v_0(\theta) \leq v \leq v_0(\theta)$ and $f(\epsilon) = 0$ otherwise. This is similar to the Fermi distribution at $T = 0$ describing cold white dwarf stars in astrophysics [43]. This is also called the water-bag distribution in plasma

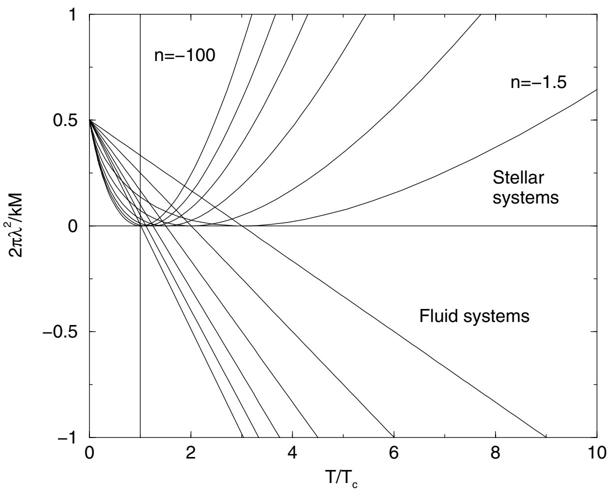

Fig. 14. Growth rate and decay rate for stellar polytropes and polytropic stars in the framework of the HMF model. The index goes from $n = -1.5$ to $n = -100$. The critical temperature is larger than for an isothermal gas ($T_n > T_c$).

physics (when v_0 is independent on θ). The density and the pressure are given by $\rho = 2\eta_0 v_0$ and $p = (2/3)\eta_0 v_0^3$. This leads to a polytropic equation of state $p = K\rho^3$ of index $n = 1/2$ and polytropic constant $K = 1/(12\eta_0^2)$. For a homogeneous system, we have the relation $M = 4\pi\eta_0 v_0$. Then, combining the preceding relations, we find that the velocity of sound is $c_s = v_0$. Therefore, the system is formally nonlinearly dynamically stable if

$$v_0^2 \leq \frac{kM}{4\pi}, \tag{163}$$

and linearly dynamically unstable otherwise. Noting that the kinetic temperature is $T = v_0^2/3$, we check that the above result returns (98) with $\gamma = 3$. Thus, $\eta_{crit} = 3$ and $\epsilon_{crit} = 1/3$. Once again, these results have been obtained with almost no calculation. This is an advantage of formula (115) with respect to formula (116).

On the other hand, using $f'(v) = \eta_0[\delta(v + v_0) - \delta(v - v_0)]$, the dielectric function (117) can be written

$$\epsilon(1, \omega) = 1 - \frac{T_{1/2}}{T} W_{1/2}(\omega/\sqrt{T}), \tag{164}$$

with

$$W_{1/2}(z) = \frac{1}{1 - \frac{1}{3}z^2}. \tag{165}$$

We look for solutions of the dispersion relation $\epsilon(1, \omega) = 0$ in the form $\omega = \Omega$ where Ω is real. This solution only exists for $T > T_{1/2}$ and corresponds to an oscillatory solution $\delta f \sim e^{i\Omega t}$. The pulsation is given by

$$\Omega = \pm\sqrt{3}(T - T_{1/2})^{1/2} = \pm\sqrt{v_0^2 - \frac{kM}{4\pi}}. \tag{166}$$

We now consider the case $\omega = i\lambda$ where λ is real. This solution only exists for $T < T_{1/2}$ and

$$\lambda = \pm\sqrt{3}(T_{1/2} - T)^{1/2} = \pm\sqrt{\frac{kM}{4\pi} - v_0^2}. \tag{167}$$

The case $\lambda > 0$ corresponds to a growing (unstable) mode $\delta f \sim e^{\lambda t}$ and the case $\lambda = -\gamma < 0$ corresponds to a damped mode $\delta f \sim e^{-\gamma t}$. We note that for this special case $n = 1/2$, the growth rate and the pulsation period of the stellar system are the same as for the corresponding barotropic gas, see equation (87). The results (166) and (167) have been previously derived by Choi and Choi [34]. They are recalled here for sake of completeness and because we will need them in Section 5.3.

5 Collisional relaxation of stellar systems

The Vlasov equation (104) can be obtained from the BBGKY hierarchy, issued from the Liouville equation (2), by using the mean-field approximation (4) which is valid in the limit $N \to +\infty$ with η and ϵ fixed. We would like now to take into account the effect of correlations between particles in order to describe the "collisional" relaxation. We shall develop a kinetic theory which takes into account terms of order $1/N$ in the correlation function.

5.1 The evolution of the whole system: the Landau equation

There are different methods to obtain a kinetic equation for the distribution function $f(\theta, v, t)$. One possibility is to start from the N-body Liouville equation and use projection operator technics. This method has been followed by Kandrup [44] for stellar systems and by Chavanis [45] for the point vortex gas. We shall first consider an application of this theory to the HMF model (Chavanis [46]). In the large N limit and neglecting collective effects, the projection operator formalism leads to a kinetic equation of the form

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \theta} + \langle F \rangle \frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\int_0^t d\tau \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t) \times\left\{\mathcal{F}(1 \to 0, t-\tau)\frac{\partial}{\partial v} + \mathcal{F}(0 \to 1, t-\tau)\frac{\partial}{\partial v_1}\right\} \times f(\theta_1, v_1, t-\tau) f(\theta, v, t-\tau). \tag{168}$$

Here, $f(\theta, v, t) = NP_1(\theta, v, t)$ is the distribution function, $\langle F \rangle(\theta, t)$ is the (smooth) mean-field force and $\mathcal{F}(1 \to 0, t) = F(1 \to 0, t) - \langle F \rangle(\theta, t)$ is the fluctuating force created by particle 1 (located at θ_1, v_1) on particle 0 (located at θ, v) at time t. Between t and $t - \tau$, the particles are assumed to follow the trajectories determined by the slowly evolving mean-field $\langle F \rangle(\theta, t)$. Equation (168) is a non-Markovian integrodifferential equation. We insist on the fact that this equation is valid for an inhomogeneous system while the kinetic equations presented below will only apply to homogeneous systems. Unfortunately, equation (168) remains too complicated for practical purposes and we will have to make simplifications. If we consider a spatially homogeneous system for which the distribution function $f = f(v, t)$ depends only on the velocity, and if we implement a Markovian approximation, the foregoing equation reduces to

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial v}\int_0^{+\infty}d\tau\int d\theta_1 dv_1 F(1\to 0,t)$$
$$\times F(1\to 0,t-\tau)\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v_1}\right)f(v_1,t)f(v,t),\quad (169)$$

where $F(1\to 0,t)=-\frac{k}{2\pi}\sin(\theta(t)-\theta_1(t))$. We thus need to calculate the memory function

$$M=\int_0^{+\infty}d\tau\int d\theta_1 F(1\to 0,t)F(1\to 0,t-\tau)$$
$$=\frac{k^2}{4\pi^2}\int_0^{+\infty}d\tau\int d\theta_1\sin(\theta-\theta_1)\sin[\theta(t-\tau)-\theta_1(t-\tau)],\quad (170)$$

where $\theta_i(t-\tau)$ is the position at time $t-\tau$ of the ith particle located at $\theta_i=\theta_i(t)$ at time t. Since the system is homogeneous, the mean force acting on a particle vanishes and the average equations of motion are $\theta(t-\tau)=\theta-v\tau$ and $\theta_1(t-\tau)=\theta_1-v_1\tau$. Thus, we get

$$M=\frac{k^2}{4\pi^2}\int_0^{+\infty}d\tau\int_0^{2\pi}d\phi\sin\phi\sin(\phi-u\tau),\quad (171)$$

where $\phi=\theta-\theta_1$ and $u=v-v_1$. The integration yields

$$M=\frac{k^2}{4\pi}\int_0^{+\infty}d\tau\cos(u\tau)=\frac{k^2}{4}\delta(u).\quad (172)$$

Therefore, the kinetic equation (169) becomes

$$\frac{\partial f}{\partial t}=\frac{k^2}{4}\frac{\partial}{\partial v}\int dv_1\delta(v-v_1)\left(f_1\frac{\partial f}{\partial v}-f\frac{\partial f_1}{\partial v_1}\right)=0.\quad (173)$$

This equation can be considered as the counterpart of the Landau equation describing the "collisional" evolution of stellar systems such as globular clusters (in that case, the system is not homogeneous but the collision term is often calculated by making a local approximation). The Landau collision term can also be obtained from the BBGKY hierarchy at the order $O(1/N)$ by neglecting the cumulant of the three-body distribution function of order $1/N^2$ [17]. For the HMF model, and for one dimensional systems in general, we find that the Landau collision term vanishes. This is because the diffusion term (first term in the r.h.s.) is equally balanced by the friction term (second term in the r.h.s.). A similar cancellation of the collision term at order $1/N$ is found in the case of 2D point vortices when the profile of angular velocity is monotonic (Dubin and O'Neil [47], Chavanis [45], Dubin [48]). Therefore, after a phase of violent relaxation, the system can remain frozen in a stationary solution of the Vlasov equation for a very long time, larger than Nt_D. Only non-trivial three-body correlations can induce further evolution of the system. However, their effect is difficult to estimate. Note that the collision term of order $1/N$ may not cancel out in the case of inhomogeneous systems and for the multi-species HMF model (see Sect. 7).

5.2 The evolution of a test particle in a thermal bath: the Fokker-Planck equation

Equations (168) and (169) can also be used to describe the evolution of the distribution function $P(v,t)$ of a test particle evolving in a bath of field particles with static distribution function $f_1(v_1)$. In that case, we have to consider that the distribution function of the bath is *given*, i.e. $f_1=f_0(v_1)$. The evolution of $P(v,t)$ is then governed by the equation

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v}\int_0^{+\infty}d\tau\int d\theta_1 dv_1 F(1\to 0,t)$$
$$\times F(1\to 0,t-\tau)\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v_1}\right)f_0(v_1)P(v,t).\quad (174)$$

Equation (174) can be written in the form of a Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v}\left(D\frac{\partial P}{\partial v}+P\eta\right).\quad (175)$$

The two terms of this equation correspond to a diffusion and a friction. The diffusion coefficient is given by the Kubo formula

$$D=\int_0^{+\infty}d\tau\langle F(t)F(t-\tau)\rangle d\tau.\quad (176)$$

The friction can be understood physically by developing a linear response theory. It arises from the response of the field particles to the perturbation induced by the test particle, as in a polarization process (see Kandrup [49]).

Introducing the memory function (172), the Fokker-Planck equation (174) can be rewritten

$$\frac{\partial P}{\partial t}=\frac{k^2}{4}\frac{\partial}{\partial v}\int dv_1\delta(v-v_1)\left(\frac{\partial}{\partial v}-\frac{\partial}{\partial v_1}\right)f_0(v_1)P(v,t).\quad (177)$$

It can be put in the form (175) where the coefficients of diffusion and friction are explicitly given by [46]:

$$D=\frac{k^2}{4}\int dv_1\delta(v-v_1)f_0(v_1)=\frac{k^2}{4}f_0(v),\quad (178)$$

$$\eta=-\frac{k^2}{4}\int dv_1\delta(v-v_1)\frac{df_0}{dv}(v_1)=-\frac{k^2}{4}f_0'(v).\quad (179)$$

More precisely, comparing equation (175) with the general Fokker-Planck equation

$$\frac{\partial P}{\partial t}=\frac{1}{2}\frac{\partial^2}{\partial v^2}\left(P\frac{\langle(\Delta v)^2\rangle}{\Delta t}\right)-\frac{\partial}{\partial v}\left(P\frac{\langle\Delta v\rangle}{\Delta t}\right).\quad (180)$$

we find that

$$\frac{\langle(\Delta v)^2\rangle}{\Delta t}=2D,\qquad\frac{\langle\Delta v\rangle}{\Delta t}=\frac{dD}{dv}-\eta.\quad (181)$$

Using equations (178) and (179), we obtain

$$\eta = -\frac{1}{2}\frac{\langle \Delta v\rangle}{\Delta t}. \tag{182}$$

Therefore, η represents half the friction force. This is the same result as in the case of Coulombian or Newtonian interactions (see, e.g., Chavanis [50]). Using equations (178) and (179), the Fokker-Planck equation (175) can be rewritten

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left[D\left(\frac{\partial P}{\partial v} - P\frac{d\ln f_0}{dv}\right)\right], \tag{183}$$

with the initial condition $P(v, t = 0) = \delta(v - v_0)$. It describes the evolution of a test particle in a potential $U(v) = -\ln f_0(v)$ created by the field particles. A similar equation is found for 2D point vortices in position space where the friction is replaced by a drift [45]. When $f_0(v)$ is the Maxwellian (18), corresponding to a statistical equilibrium state (thermal bath approximation), equation (183) takes the form of the Kramers equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left[D(v)\left(\frac{\partial P}{\partial v} + \beta P v\right)\right], \tag{184}$$

as in the theory of Brownian motion [51]. We note in particular that the friction coefficient is given by the Einstein formula $\xi = D\beta$. However, in the present context, the diffusion coefficient depends on the velocity and, in the ballistic approach that we have considered, is given by [46]:

$$D(v) = \frac{\rho k^2}{4}\left(\frac{\beta}{2\pi}\right)^{1/2} e^{-\beta\frac{v^2}{2}}. \tag{185}$$

We note that $P(v, t)$ always converges to the distribution function of the bath $NP(v, t) \to f_0(v)$ for $t \to +\infty$ while for $3D$ self-gravitating system, this is the case only when f_0 is the statistical equilibrium distribution.

Finally, again neglecting collective effects, a simple calculation shows that the temporal correlations of the force are

$$\langle F(0)F(t)\rangle = \frac{k^2}{4\pi}\int_{-\infty}^{+\infty}\cos[(v-v_1)t] f_0(v_1)dv_1 = \frac{k^2}{2}\cos(vt)\hat{f}_0(t) = 2\cos(vt)\hat{D}(t), \tag{186}$$

where $\hat{D}(t)$ is the Fourier transform of $D(v)$. For the Maxwellian distribution, we get

$$\langle F(0)F(t)\rangle = \frac{\rho k^2}{4\pi}\cos(vt)e^{-\frac{t^2}{2\beta}}, \tag{187}$$

which seems to indicate a Gaussian decay of the correlations.

5.3 Collective effects: the Lenard-Balescu equation

The kinetic theory developed previously (Chavanis [46]), while useful as a first step, is however inaccurate because it is based on a ballistic approximation and ignores collective effects which are non-negligible for the HMF model close to the critical temperature (Bouchet [52]). In the case of 3D stellar systems, collective effects have only a weak influence on the kinetic theory and they are often neglected. This implicitly assumes that the size of the system is smaller than the Jeans length (recall that the Jeans length plays the role of the critical temperature in the present context). In general, collective effects can be taken into account by developing a kinetic theory as in the case of plasmas [35]. Noting that the HMF model is the one Fourier mode of a one dimensional plasma, Inagaki [9] proposed to describe the collisional evolution of the system by the corresponding form of the Lenard-Balescu equation. It can be written

$$\frac{\partial f}{\partial t} = \frac{k^2}{4}\frac{\partial}{\partial v}\int dv_1 \frac{\delta(v-v_1)}{|\epsilon(1,v)|^2}\left(f_1\frac{\partial f}{\partial v} - f\frac{\partial f_1}{\partial v_1}\right) = 0, \tag{188}$$

where $\epsilon(1, v)$ is the dielectric function (117). We note that the collision term again cancels out. However, if we use this equation to describe the evolution of a test particle in a thermal bath, as we did in Section 5.2 by replacing f_1 by $f_0(v_1)$, we obtain equation (183) where the expression of the diffusion coefficient is now given by

$$D = \frac{k^2}{4}\frac{f_0(v)}{|\epsilon(1,v)|^2}. \tag{189}$$

It differs from the preceding expression (178) due to the occurrence of the term $|\epsilon(1, v)|^2$ in the denominator which takes into account collective effects. For the Maxwellian distribution, the dielectric function is given by equations (128) and (130). This leads to the expression of the diffusion coefficient

$$D(v) = \frac{\frac{\rho k^2}{4}(\frac{\beta}{2\pi})^{1/2}e^{-\beta\frac{v^2}{2}}}{[1-\eta A(\sqrt{\beta}v)]^2 + \frac{\pi}{2}\eta^2\beta v^2 e^{-\beta v^2}} \tag{190}$$

with $A(x) = 1 - xe^{-\frac{x^2}{2}}\int_0^x e^{\frac{u^2}{2}}du$. We note that $A(x) = 1 - x^2 + ...$ for $x \to 0$ and $A(x) \sim -1/x^2$ for $x \to +\infty$. Therefore, the diffusion coefficient behaves as equation (185) for $v \to +\infty$ and tends to $\frac{\rho k^2}{4}(\beta/2\pi)^{1/2}/(1-\eta)^2$ for $v \to 0$ and $\eta < 1$. At the critical temperature $\eta = 1$ it diverges as $D(v) \sim \frac{\rho k^2}{2\pi}(\beta/2\pi)^{1/2}/(\beta v^2)$ for $v \to 0$. Its behaviour is represented in Figure 15.

The expression (190) of the diffusion coefficient was obtained by Bouchet [52] in a different manner, by analyzing the stochastic process of equilibrium fluctuations. This approach was then generalized to an arbitrary distribution function by Bouchet and Dauxois [53], leading to equation (189). In fact, formulae (189) and (190) correspond to the one dimensional version of the diffusion coefficient computed in plasma physics [35] when the potential of interaction is truncated to one Fourier mode. The general expression of the diffusion coefficient and of the Fokker-Planck equation is given in Chavanis [17] for an arbitrary potential of interaction in D dimensions.

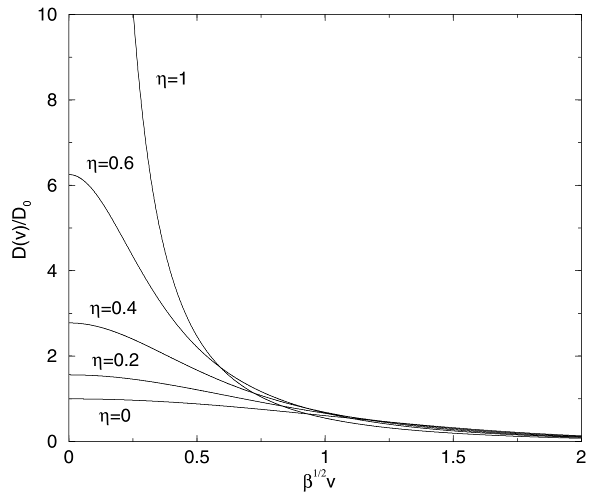

Fig. 15. Dependence of the diffusion coefficient $D(v)$ on the velocity v for different values of the inverse temperature $\eta = \beta/\beta_c$. The normalization constant is $D_0 = \frac{nk^2}{4}(\beta/2\pi)^{1/2}$ corresponding to $D(0)$ for $\eta = 0$. We note that $D(0)$ increases as η increases and that it diverges at the critical point $\eta = 1$.

We emphasize that the previous results are valid for an arbitrary distribution function $f_0(v)$ of the bath provided that it is stable with respect to the Vlasov equation. This is because, as we shall see, the relaxation time of the "field particles" (bath) towards statistical equilibrium (Maxwellian) is longer than the relaxation time of a "test particle" towards f_0, so that the distribution function f_0 of the bath can be considered as "frozen". The general expression of the diffusion coefficient can be written

$$D(v) = \frac{\frac{k^2}{4} f_0(v)}{[1 + \frac{k}{2}\mathcal{P}\int_{-\infty}^{+\infty} \frac{f_0'(w)}{w-v} dw]^2 + \frac{k^2\pi^2}{4} f_0'(v)^2}, \quad (191)$$

where $\mathcal{P}$ stands for the principal value. Its asymptotic behaviour for $v \to +\infty$ is always given by equation (178). As a complementary example to Figure 15, we have plotted in Figures 16 and 17 the diffusion coefficient corresponding to a polytropic distribution of index $n = 1$ and $n = -2$ (in that case f_0 is given by equation (153) where ρ and T are uniform). Another example is provided by the water-bag model for which an explicit expression of $D(v)$ can be given. Using equations (164), (165) and (166), it can be written conveniently as

$$D(v) = \frac{k^2}{4}\eta_0 \left[\frac{v_0^2 - v^2}{\Omega^2 - v^2}\right]^2, \quad (192)$$

for $v \leq v_0$ and $D = 0$ otherwise. The diffusion coefficient diverges when the velocity of the particle v is in phase with the frequency $\Omega < v_0$ of the wave arising from the slightly perturbed distribution of field particles (see Sect. 4.7.3). This divergence occurs because, for the water bag distribution, there exists purely oscillatory modes for a wide range of temperatures. In general, the diffusion coefficient (189) diverges only at the critical point for $v = \omega_r$ with $f'(\omega_r) = 0$; this precisely correspond to the criterion

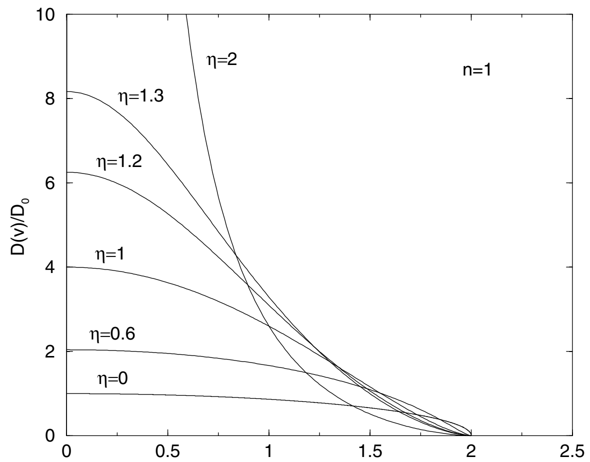

Fig. 16. Same as Figure 15 for a polytropic distribution with index $n = 1$. Here $\eta = kM/4\pi T$ where T is the kinetic temperature. In this case, the critical point is $\eta_1 = 2$. The diffusion coefficient vanishes at the maximum velocity $v_{max} = \sqrt{4T}$.

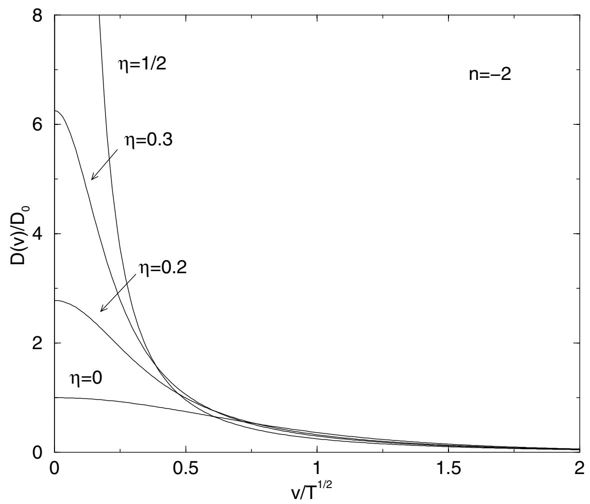

Fig. 17. Same as Figure 15 for a polytropic distribution with index $n = -2$. Here $\eta = kM/4\pi T$ where T is the kinetic temperature. In this case, the critical point is $\eta_{-2} = 1/2$. This figure is very similar to Fig. 15 except that the diffusion coefficient decreases algebraically.

of marginal stability (120). For example, for the Gaussian distribution, we recover the divergence at $T = T_c$ for $v = \omega_r = 0$.

5.4 The auto-correlation function

We note that for high temperatures (i.e., $\eta \to 0$), equation (190) reduces to the expression (185) found in the ballistic approach developed in Sect. 5.2. This is because, in that limit, collective effects are weak with respect to the pure ballistic motion and $\epsilon \simeq 1$ according to equation (128). However, the behaviour of the correlation function is different. Indeed, a direct analysis [17] shows that the temporal auto-correlation function of the force is given by

$$\langle F(0)F(t)\rangle = \frac{k^2}{4\pi}\int_{-\infty}^{+\infty}\frac{\cos[(v-v_1)t]}{|\epsilon(1,v_1)|^2}f_0(v_1)dv_1 = 2\cos(vt)\hat{D}(t). \quad (193)$$

Substituting this expression in the Kubo formula (176) and using $\delta(v-v_1) = \frac{1}{\pi}\int_0^{+\infty}\cos[(v-v_1)t]dt$, we recover equation (189). In the ballistic approximation, the correlation function (187) is Gaussian while the exact treatment taking into account collective effects shows that the decay of the fluctuations is in fact exponential with a decay exponent $\gamma(\beta) = (2/\beta)^{1/2}F^{-1}(\eta)$ where $F(x) = 1/(1+\sqrt{\pi}xe^{x^2}\mathrm{erfc}(-x))$ is the F-function (138). This result was obtained by Bouchet [52] by working out the integro-differential equation satisfied by the autocorrelation function. We shall present here an alternative derivation (Chavanis [17]) which will make a clear link with the decay exponent appearing in the linear stability analysis of the Vlasov equation in Section 4.7.1. Noting that the correlation function is proportional to the Fourier transform of the diffusion coefficient, according to equation (193), we can obtain the expression of γ by determining the pole of $D(v)$ in equation (189). Setting $v = i\lambda$ where λ is real, we find after some calculations that

$$|\epsilon|^2(1,i\lambda) = \epsilon(1,i\lambda)\epsilon(1,-i\lambda) \quad (194)$$

where we recall that $\epsilon(1,i\lambda)$ is given by equation (131). Clearly, $|\epsilon|^2(1,i\lambda)$ is an even function of λ. We need to determine the values of λ for which this function vanishes. Since $\eta < 1$, we find that $\lambda = \pm\gamma$ where $\gamma > 0$ is determined by $\epsilon(1,-i\gamma) = 0$. Therefore, γ is the damping rate of the stable perturbed solutions of the Vlasov equation; it is given by equation (137). Next, we consider $\lambda = \pm\gamma + \epsilon$. Expanding equation (194) for $\epsilon \ll 1$, we find after elementary calculations that

$$D(v) \sim \frac{K(\gamma)}{v^2+\gamma^2}, \quad K(\gamma) = \frac{2T}{\sqrt{\pi}}\frac{1}{|F'(\frac{\gamma}{\sqrt{2T}})|} \quad (195)$$

for $v \to \pm i\gamma$. Therefore, for $t \to +\infty$, the correlation function (193) is the Fourier transform of a Lorentzian so it decays like

$$\langle F(0)F(t)\rangle \sim \frac{k^2 M}{8\pi^2}\frac{\sqrt{2T}}{\gamma}\frac{1}{|F'(\frac{\gamma}{\sqrt{2T}})|}\cos(vt)e^{-\gamma t}, \quad (196)$$

with

$$\gamma = (2/\beta)^{1/2}F^{-1}(\eta). \quad (197)$$

The exponential decay of the correlation function corresponds to the Markovian limit of the stochastic process, thereby justifying the Markovian approximation in the kinetic theory. This is quite different from the correlations of the gravitational force which decay as t^{-1} (Chandrasekhar [54]). This slow decay may throw doubts on the validity of the ordinary Landau equation, based on a Markovian approximation, used to describe stellar systems (see Kandrup [44] for a detailed discussion). For the HMF model, the decay exponent $\gamma(\beta)$ depends on the temperature. It diverges like $\gamma \sim \sqrt{2T\ln T}$ as $T \to +\infty$. This is why the correlation function is Gaussian in the treatement neglecting collective effects (see Sect. 5.2). On the other hand, $\gamma \sim (8/kM)^{1/2}(T-T_c)$ as $T \to T_c^+$, so that the correlation function decreases very slowly close to the critical temperature. This may invalidate the Markovian approximation close to the critical point. The Fokker-Planck equation (184) with the diffusion coefficient (190) has been recently investigated by Bouchet & Dauxois [53]. In this paper, using the rapid decay for large v of the diffusion coefficient (190), the numerically observed [55,56] anomalous algebraic decay of the velocity autocorrelation function is explained and algebraic exponents are explicitly computed. For a large class of bath distribution function f_0, this may also lead to anomalous diffusion of angles θ.

5.5 The relaxation timescale

Let us consider the relaxation of a test particle in a thermal bath. Due to the rapid decrease of $D(v)$ for large v, the spectrum of the Fokker-Planck equation (184) has no gap, and it exists no exponential relaxation time (see Bouchet and Dauxois [53] for a detailed discussion). A time scale will however describe relative relaxation speeds for different values of the temperature T. We shall obtain an estimate of this time scale. Ignoring collective effects in a first step, this process is described by equations (184–185). If the diffusion coefficient were constant, we would deduce that the dispersion of the particles increases as $\langle(\Delta v)^2\rangle = 2Dt$. Introducing the r.m.s velocity $v_m = \langle v^2\rangle^{1/2}$, we define the relaxation timescale t_r such that $\langle(\Delta v)^2\rangle = v_m^2$. This leads to $t_r = v_m^2/2D$. Since D depends on v, the description of the diffusion process is more complex. However, the formula

$$t_r = \frac{v_m^2}{2D(v_m)}, \quad (198)$$

provides a useful estimate of the speed of relaxation. For the Maxwellian distribution for which $v_m = 1/\sqrt{\beta}$, we get

$$t_r = \frac{v_m^3}{0.121\rho k^2}. \quad (199)$$

We can also estimate the relaxation timescale by $t_r' = 1/\xi$, where ξ is the friction coefficient. Using the Einstein relation $\xi = D\beta$, this yields $t_r' = 2t_r$. Finally, setting $w = v/(\sqrt{2}v_m)$, we can rewrite equation (184) in the dimensionless form

$$\frac{\partial P}{\partial t} = \frac{1}{t_R}\frac{\partial}{\partial w}\left[G(w)\left(\frac{\partial P}{\partial w} + 2Pw\right)\right], \quad (200)$$

where $G(w) = e^{-w^2}$ and

$$t_R = \frac{v_m^3}{0.05\rho k^2}, \quad (201)$$

which provides a useful time renormalization factor in the Fokker-Planck equation. If we use expression (190) of the diffusion coefficient, we need to multiply the relaxation timescale by $|\epsilon(1, v_m)|^2 = (1 + 2.25\eta)^2 + 0.578\eta^2$. In particular, close to the critical temperature (i.e. $\eta = 1$), the relaxation timescale is multiplied by $\simeq 11.1$.

If we now introduce a dynamical timescale through the relation

$$t_D = \frac{2\pi}{v_m}, \tag{202}$$

we find that $t_R/t_D \sim v_m^4/Nk^2 \sim 1/Nk^2\beta^2 \sim N/\eta^2 \sim N$ in the thermodynamic limit of Section 2.1. Therefore, the relaxation time of a test particle in a thermal bath increases linearly with the number N of field particles. More precisely, we have

$$t_R = \frac{0.127}{\eta^2} N t_D. \tag{203}$$

The collision term in equation (200) is of order $1/N$ in the thermodynamic limit $N \to +\infty$ with η fixed. This scaling also holds for the collision term in the Landau equation (173) noting that $k \sim 1/N$ and $f \sim N$ if we take $v_m \sim 1$. It represents therefore the first correction to the Vlasov limit in an expansion in $1/N$ of the correlation function. However, contrary to the relaxation of a single particle in a thermal bath, the relaxation time of the whole system is *not* $t_{relax} \sim Nt_D$ because, as we have seen, the collision term cancels out at the order $1/N$. Therefore, the relaxation time of the whole system is larger than Nt_D. This is consistent with the finding of Yamaguchi et al. [16] who numerically obtain $t_{relax} \sim N^{1.7} t_D$.

It is interesting to compare this result with other systems with long-range interactions. For stellar systems, the Chandrasekhar relaxation time scales as $t_{relax} \sim \frac{N}{\ln N} t_D$. It corresponds to the Nt_D scaling polluted by logarithmic corrections. This is the relaxation time of a test star immersed in a bath of field stars as well as the relaxation time of the whole cluster itself (in the absence of gravothermal catastrophe). For the point vortex gas, the collision term in the kinetic equation cancels out at the order $1/N$ when the profile of angular velocity is monotonic. Therefore, the relaxation time of the whole system is larger than Nt_D. In fact, it is not clear whether the point vortex gas ever relaxes towards statistical equilibrium [3,48]. By contrast, the relaxation time of a test vortex in a thermal bath of field vortices scales as $t_{relax} \sim \frac{N}{\ln N} t_D$ [3].

6 Self-attracting Brownian particles: the BMF model

The Hamilton equations (1) describe an isolated system of particles with long-range interactions. Since energy is conserved, the fundamental statistical description of the system is based on the microcanonical ensemble. It can be of interest to consider in parallel the case where the system is stochastically forced by an external medium. We thus introduce a system of Brownian particles with long-range interactions which is the canonical version of the Hamiltonian system (1). This could be called the BMF (Brownian Mean Field) model. Likewise in the case of 3D Newtonian interactions, a system of self-gravitating Brownian particles has been recently introduced and studied (see Chavanis et al. [57] and subsequent papers).

6.1 Non-local Kramers and Smoluchowski equations

We consider a one-dimensional system of self-attracting Brownian particles with cosine interaction whose dynamics is governed by the N-coupled stochastic equations

$$\frac{d\theta_i}{dt} = v_i,$$
$$\frac{dv_i}{dt} = -\frac{\partial}{\partial\theta_i} U(\theta_1, ..., \theta_N) - \xi v_i + \sqrt{2D} R_i(t), \tag{204}$$

where $-\xi v_i$ is a friction force and $R_i(t)$ is a white noise satisfying

$$\langle R_i(t) \rangle = 0, \qquad \langle R_i(t) R_j(t') \rangle = \delta_{ij}\delta(t - t'), \tag{205}$$

where $i = 1, ..., N$ refer to the particles. The particles interact through the potential $U(\theta_1, ..., \theta_N) = \sum_{i<j} u(\theta_i - \theta_j)$ where $u(\theta_i - \theta_j) = -\frac{k}{2\pi}\cos(\theta_i - \theta_j)$. We define the inverse temperature $\beta = 1/T$ through the Einstein relation $\xi = D\beta$. The stochastic model (204–205) is analogous to the model of self-gravitating Brownian particles introduced by Chavanis et al. [57]. For this system, the relevant ensemble is the canonical ensemble where the temperature measures the strength of the stochastic force. The evolution of the N-body distribution function is governed by the N-body Fokker-Planck equation

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N}\left(v_i \frac{\partial P_N}{\partial \theta_i} + F_i \frac{\partial P_N}{\partial v_i}\right) = \sum_{i=1}^{N} \frac{\partial}{\partial v_i}\left(D \frac{\partial P_N}{\partial v_i} + \xi P_N v_i\right), \tag{206}$$

where $F_i = -\frac{\partial U}{\partial \theta_i}$. The stationary solution corresponds to the canonical distribution

$$P_N = \frac{1}{Z} e^{-\beta(\sum_{i=1}^{N} \frac{v_i^2}{2} + U(\theta_1, ..., \theta_N))}. \tag{207}$$

We note that the canonical distribution (207) is the *only* stationary solution of the N-body Fokker-Planck equation while the microcanonical distribution (3) is just a *particular* stationary solution of the Liouville equation (see Sect. 2.1). For the system (204–205), the equilibrium canonical distribution does not rely, therefore, on a postulate. In the thermodynamic limit $N \to +\infty$ with $\eta = \beta k M/4\pi$ fixed, one can prove that the N particles distribution function factorizes and that the mean-field approximation is exact [58,17]. The evolution of the one particle distribution function is then governed by the non-local Kramers equation

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial \theta}-\frac{\partial \Phi}{\partial \theta}\frac{\partial f}{\partial v}=\frac{\partial}{\partial v}\left(D\frac{\partial f}{\partial v}+\xi f v\right), \qquad (208)$$

which has to be solved in conjunction with equation (8). These equations have been considered, independently, by Choi and Choi [34].

To simplify the problem further, we shall consider a strong friction limit $\xi\rightarrow +\infty$ or, equivalently, a long time limit $t\gg \xi^{-1}$. In that case, we can neglect the inertia of the particles and the stochastic equations (204) reduce to

$$\frac{d\theta_i}{dt}=-\mu\frac{\partial}{\partial\theta_i}U(\theta_1,...,\theta_N)+\sqrt{2D_*}R_i(t), \qquad (209)$$

where $\mu=1/\xi$ is the mobility and $D_*=D/\xi^2=T/\xi$ is the diffusion coefficient in physical space. The evolution of the N-body distribution function is governed by the N-body Fokker-Planck equation

$$\frac{\partial P_N}{\partial t}=\sum_{i=1}^{N}\frac{\partial}{\partial \theta_i}\left[D_*\frac{\partial P_N}{\partial\theta_i}+\mu P_N\frac{\partial}{\partial\theta_i}U(\theta_1,...,\theta_N)\right]. \qquad (210)$$

The stationary solution corresponds to the canonical distribution in configuration space

$$P_N=\frac{1}{Z}e^{-\beta U(\theta_1,...,\theta_N)}. \qquad (211)$$

In the thermodynamic limit $N\rightarrow +\infty$ with $\eta=\beta kM/4\pi$ fixed, the mean-field approximation is exact and the evolution of the one particle distribution function is governed by the non-local Smoluchowski equation

$$\frac{\partial \rho}{\partial t}=\frac{\partial}{\partial\theta}\left[\frac{1}{\xi}\left(T\frac{\partial\rho}{\partial\theta}+\rho\frac{\partial\Phi}{\partial\theta}\right)\right], \qquad (212)$$

where Φ is given by equation (8). Alternatively, the Smoluchowski equation (212) can be obtained from the non-local Kramers equation (208) by using a Chapman-Enskog expansion in power of $1/\xi$ [59]. In that approximation, the distribution function is close to the Maxwellian

$$f(\theta,v,t)=\frac{1}{\sqrt{2\pi T}}\rho(\theta,t)e^{-\frac{v^2}{2T}}+O(\xi^{-1}), \qquad (213)$$

and the evolution of the density is governed by the non-local Smoluchowski equation. The equations (208) and (212) conserve mass and decrease the Boltzmann free energy, i.e. $\dot F_B\le 0$. This is the canonical version of the H-theorem [31,50].

6.2 Linear stability

The stationary solutions of equation (212) are given by equation (19). They extremize the Boltzmann free energy $F_B=E-TS_B$ with (45) and (46) at fixed mass and temperature. Furthermore, only stationary solutions that *minimize* the free energy are linearly dynamically stable with respect to the non-local Smoluchowski equation [31]. Therefore, thermodynamical and dynamical stability are clearly connected: the stable stationary solutions of equation (212) correspond to the canonical statistical equilibrium states in the mean-field approximation.

Considering a small perturbation $\delta\rho$ around a stationary solution of equation (212), we get

$$\frac{\partial\delta\rho}{\partial t}=\frac{\partial}{\partial\theta}\left[\frac{1}{\xi}\left(T\frac{\partial\delta\rho}{\partial\theta}+\delta\rho\frac{\partial\Phi}{\partial\theta}+\rho\frac{\partial\delta\Phi}{\partial\theta}\right)\right]. \qquad (214)$$

Writing $\delta\rho\sim e^{\lambda t}$ and introducing the notation (50), we obtain the eigenvalue equation

$$\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{dq}{d\theta}\right)+\frac{k}{2\pi T}\int_0^{2\pi}q(\theta')\cos(\theta-\theta')d\theta'=\frac{\lambda\xi}{T\rho}q, \qquad (215)$$

which is similar to the eigenvalue equations obtained in the preceding sections. These eigenvalue equations all coincide at the point of marginal stability $\lambda=0$ implying that the onset of the instability is the same in all the models considered. Equation (215) is the counterpart of equation (27) of Chavanis et al. [57] for self-gravitating Brownian particles. We note that the eigenvalue equations (91) and (215) for the Euler model and the Brownian model only differ by the substitution $\lambda^2\rightarrow\lambda\xi$. Therefore, the results of Section 3.5.1 can be directly extended to the present context.

For the uniform phase, the most destabilizing mode ($n=1$) is

$$\delta\rho=a_1\cos\theta\, e^{\lambda t}, \qquad (216)$$

with a growth rate

$$\lambda=\frac{1}{\xi}(T_c-T),\qquad T_c=\frac{kM}{4\pi}. \qquad (217)$$

For $T<T_c$, the perturbation grows exponentially while it is damped exponentially for $T>T_c$.

Considering now the clustered phase and using a perturbative approach similar to that of Appendix A for $T\rightarrow T_c^-$ (not detailed), we find that the perturbation is damped exponentially with a rate

$$\lambda=-\frac{2}{\xi}(T_c-T). \qquad (218)$$

From equations (217) and (218), we find that, starting from a homogeneous solution at $T=T_c^-$, there is first an exponential growth on a timescale $\xi(T_c-T)^{-1}$. Then, non-linear terms come into play and the system relaxes towards a clustered state on a timescale $(1/2)\xi(T_c-T)^{-1}$.

6.3 Local Fokker-Planck equation

Before studying the non-local Smoluchowski equation (212)-(8) numerically, it may be of interest to study a simplified problem where Φ is replaced by its equilibrium

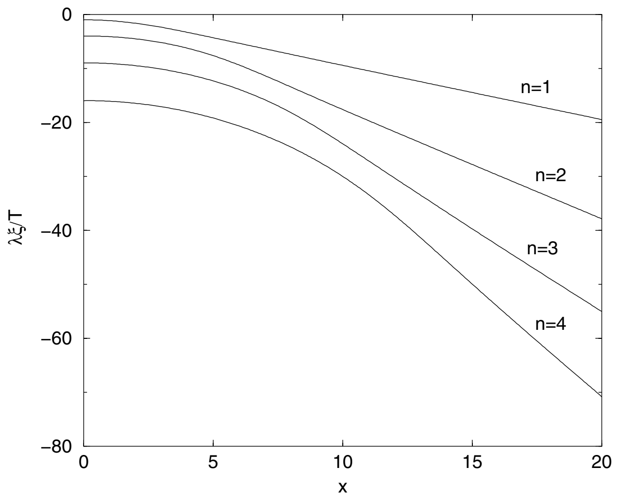

Fig. 18. Eigenvalues of the Sturm-Liouville equation (220).

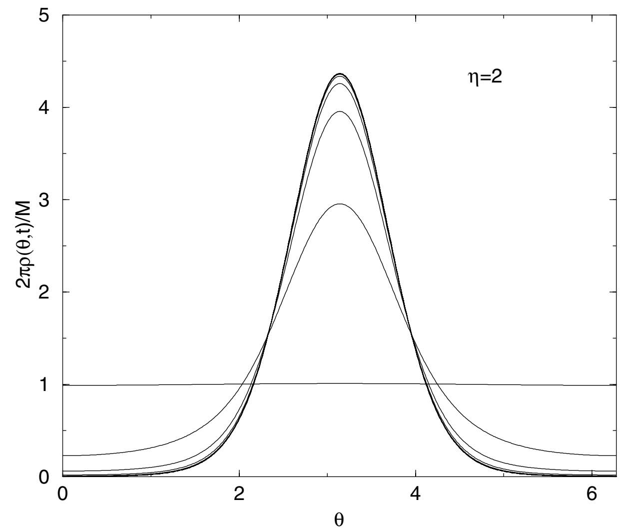

Fig. 19. Evolution of the density profile according to the local Fokker-Planck equation for $\eta = 2$ (corresponding to $x = 3.33$) starting from a homogeneous solution (numerical simulation).

value $\Phi = B\cos\theta$, where B is fixed. Considering again the strong friction limit, we obtain the classical (local) Smoluchowski equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial\theta}\left[\frac{1}{\xi}\left(T\frac{\partial\rho}{\partial\theta} - B\rho\sin\theta\right)\right]. \tag{219}$$

This Fokker-Planck equation also appears in the model of rotation of dipoles in a constant electric field developed by Debye. The stationary solutions of this equation are given by equation (19). Considering a perturbation $\delta\rho \sim e^{\lambda t}$ around a stationary solution and setting $\delta\rho = dq/d\theta$, we obtain the eigenvalue equation

$$\frac{d}{d\theta}\left(\frac{1}{\rho}\frac{dq}{d\theta}\right) = \frac{\lambda\xi}{T\rho}q. \tag{220}$$

This equation has the form of a Sturm-Liouville problem. For $B \ll 1$, so that ρ is approximately uniform, we find to leading order that the eigenvalues are $\lambda_n = -n^2T/\xi$ where $n = 1, 2, ...$ Using a procedure similar to that of Appendix A (not detailed), the next order correction is $-\lambda_n\xi/T = n^2 + n^2/[2(4n^2-1)]x^2 + ...$ where $x = \beta B \ll 1$. The eigenvalues of the Sturm-Liouville equation (220) are evaluated numerically in Figure 18.

For $B \ll 1$, the Fokker-Planck equation (219) can be solved analytically. Assuming that, initially, the density is uniform, we find

$$\rho = \frac{M}{2\pi}\left[1 + (e^{-t/\xi\beta} - 1)\beta B\cos\theta\right]. \tag{221}$$

For $t \gg t_{relax} = \xi\beta$ (relaxation time), the density reaches its equilibrium value (19). For short times $t \ll 1$, we have

$$\rho = \frac{M}{2\pi}\left(1 - \frac{1}{\xi}tB\cos\theta\right). \tag{222}$$

Starting from a homogeneous solution, there is first a linear growth followed by an exponential relaxation towards the clustered state on a timescale $\sim \xi/T$. This is illustrated numerically in Figure 26 and compared with the time evolution of the non-local Fokker-Planck equation (212). The evolution is of course quite different since, as we have seen, the non-local Fokker-Planck equation displays an exponential growth on a timescale $\xi(T_c - T)^{-1}$ followed by an exponential relaxation on a timescale $(1/2)\xi(T_c - T)^{-1}$. These timescales diverge as we approach the critical temperature T_c while there is no critical temperature for the local Fokker-Planck equation where B is fixed. The numerical solution of equation (219) is shown in Figure 19 and the analytical solution (221) in Figure 20 for two different values of η.

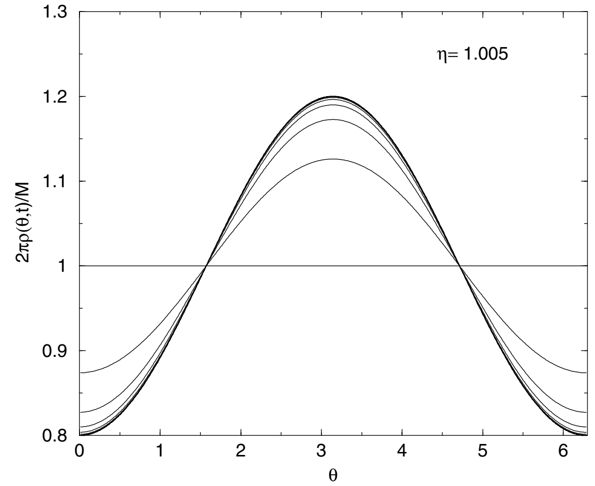

Fig. 20. Evolution of the density profile according to the local Fokker-Planck equation for $\eta = 1.005$ (corresponding to $x = 0.2$) starting from a homogeneous solution. In this case, we have used the analytical solution (221) which is valid for $B \ll 1$.

We also note that the change of variables

$$\rho = e^{-\frac{1}{2}\beta\Phi}\psi(\theta, t), \tag{223}$$

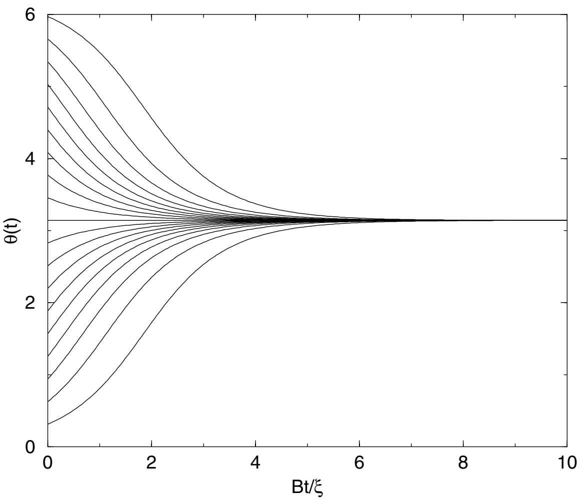

Fig. 21. Characteristics of equation (229). For $t \to +\infty$, all the particles converge at $\theta = \pi$ and a Dirac peak is formed.

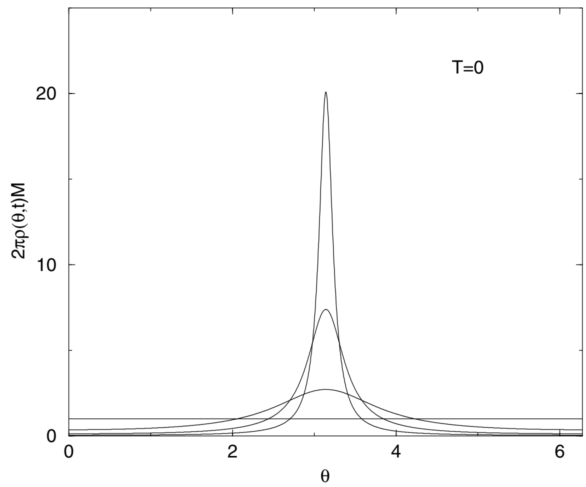

Fig. 22. Evolution of the density profile at $T = 0$ starting from a homogeneous distribution.

transforms the Fokker-Planck equation (219) into a Schrödinger-like equation (with imaginary time)

$$\xi \frac{\partial \psi}{\partial t} = T \frac{\partial^2 \psi}{\partial \theta^2} + V(\theta)\psi, \tag{224}$$

with

$$V(\theta) = \frac{1}{2}\left[\Delta\Phi - \frac{\beta}{2}(\nabla\Phi)^2\right]\psi. \tag{225}$$

In our case

$$V(\theta) = -\frac{1}{2}\left[B\cos\theta + \frac{\beta}{2}B^2\sin^2\theta\right]. \tag{226}$$

We can use this formalism to study the low temperature regime $T \to 0$. In that case, the equilibrium state is close to a Dirac peak centered on $\theta = \pi$ so that we can expand the potential to leading order in $x = \theta - \pi$. Equation (219) becomes a Kramers-like equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[\frac{1}{\xi}\left(T\frac{\partial \rho}{\partial x} + \rho B x\right)\right]. \tag{227}$$

The corresponding Schrödinger-like equation (224) is that of a harmonic oscillator

$$\xi \frac{\partial \psi}{\partial t} = T \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}(B - \frac{1}{2}\beta B^2 x^2)\psi. \tag{228}$$

Therefore, for $T \to 0$ the eigenvalues are given by $\lambda_n = -nB/\xi$, i.e. $\lambda_n \xi / T = -nx$ (with $n = 1, 2, ...$) and the corresponding eigenfunctions are the Hermite polynomials.

It is also possible to solve the problem exactly at $T = 0$ by using the method of characteristics. Indeed, the Fokker-Planck equation becomes

$$\xi \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho B \sin\theta) = 0, \tag{229}$$

which is equivalent to an advection equation by a velocity field $v = (B/\xi)\sin\theta$. Therefore, the evolution is deterministic and the equation of motion of a particle is

$$\frac{d\theta}{dt} = \frac{1}{\xi} B \sin\theta. \tag{230}$$

This equation of motion is readily solved and we get

$$\theta(t) = 2\text{Arctan}\left[\tan\left(\frac{\theta_0}{2}\right)e^{Bt/\xi}\right], \tag{231}$$

where θ_0 is the initial position of the particle. The characteristics are shown in Figure 21. For $t \to +\infty$, all the particles converge at $\theta = \pi$ and a Dirac peak is formed. The density profile is determined by the condition $\rho_0 d\theta_0 = \rho(\theta, t) d\theta$ yielding $\rho(\theta, t) = \rho_0/(d\theta/d\theta_0)$. From equation (231), we get

$$\rho(\theta, t) = \frac{\rho_0[1 + \tan^2(\theta/2)]e^{-Bt/\xi}}{1 + \tan^2(\theta/2)e^{-2Bt/\xi}}. \tag{232}$$

This profile is shown in Figure 22 at different times.

To make the link between Figures 21 and 10, we can consider an intermediate hydrodynamical equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} - \frac{\partial \Phi}{\partial \theta} - \xi u. \tag{233}$$

This could be called the damped Euler equation (see Chavanis [31]). For $\xi = 0$ we recover the Euler equation (67) and for $\xi \to +\infty$, using the continuity equation (66), we recover the Smoluchowski equation (212). For $p = 0$ and $\Phi = B\cos\theta$, equation (233) can be solved by the method of characteristics and the results are reported in Figure 23. This clearly shows the passage from Figure 10 to Figure 21 as the friction parameter ξ increases.

6.4 Dynamics of Brownian particles in interaction

We now turn to the evolution of a system of Brownian particles in interaction described by the N-coupled

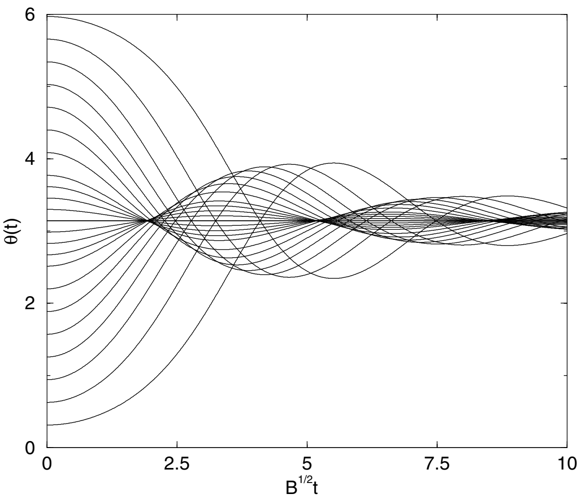

Fig. 23. Characteristics of the damped Euler equation (233) for $p = 0$ and $\Phi = B\cos\theta$. The ratio $\xi/B = 1/2$.

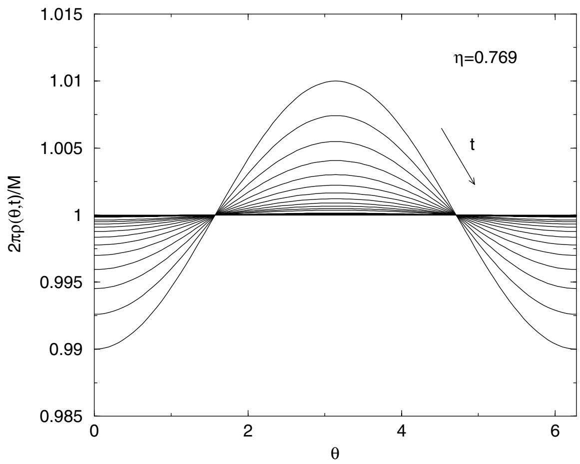

Fig. 24. Evolution of the density profile according to the non-local Fokker-Planck equation. For $\eta = 0.769 < 1$, the homogeneous solution is stable.

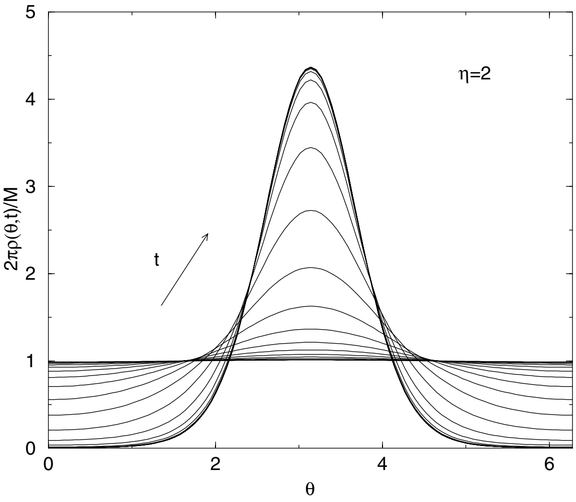

Fig. 25. For $\eta = 2 > 1$, the homogeneous solution (disk-like) is unstable and the system forms a cluster (bar-like). The evolution is longer than with the local Fokker-Planck equation represented in Figure 19 as also shown in the next figure and explained in the text.

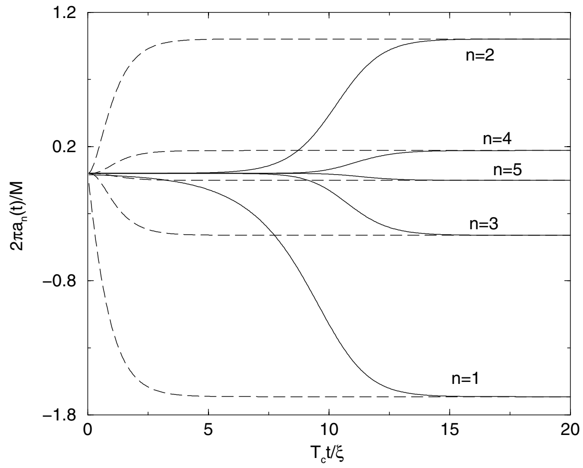

Fig. 26. Evolution of the different modes $a_n(t)$ for the non-local (full lines) and local (dashed lines) Fokker-Planck equations. The control parameters are $\eta = 2$, $x = 3.33$.

stochastic equations (204). Despite all the simplifications introduced, this model remains a non-trivial and interesting model exhibiting a process of self-organization. In particular, it shows the passage from a homogeneous phase (disk-like) to a clustered phase (bar-like) under the influence of long-range interactions. Interestingly, the large N limit of this system is exactly described by an explicit kinetic equation (208) reducing to equation (212) for large times. By contrast, the kinetic equation describing the evolution of the HMF model (1) towards statistical equilibrium is not known as the collision term cancels out at the order $1/N$ within the approximations usually considered (see Sect. 5).

For the Brownian gas, we have to solve the integro-differential equation

$$\xi\frac{\partial\rho}{\partial t} = T\frac{\partial^2\rho}{\partial\theta^2} + \frac{k}{2\pi}\frac{\partial}{\partial\theta}\left\{\rho\int_0^{2\pi}\sin(\theta-\theta')\rho(\theta',t)d\theta'\right\}. \tag{234}$$

Substituting the decomposition

$$\rho = a_0(t) + \sum_{n=1}^{+\infty} a_n(t)\cos(n\theta) \tag{235}$$

in equation (234), we find that

$$a_0 = \frac{M}{2\pi}, \tag{236}$$

$$\xi\frac{da_1}{dt} + Ta_1 = \frac{k}{2}a_1\left(\frac{M}{2\pi} - \frac{a_2}{2}\right), \tag{237}$$

$$\xi\frac{da_n}{dt} + Tn^2a_n = \frac{k}{4}na_1(a_{n-1} - a_{n+1}), \qquad (n \geq 2). \tag{238}$$

We note that the coefficient $a_1(t)$ is related to the magnetization $B(t)$, defined in equation (10), by the formula $B(t) = -\frac{k}{2}a_1(t)$. According to equations (20), (21) and (22), the coefficients a_n are given at equilibrium by

$$a_n = \frac{M}{\pi}(-1)^n \frac{I_n(\beta B)}{I_0(\beta B)}. \tag{239}$$

Therefore, the static equations (237) and (238) with $d/dt = 0$ coincide with the recursive relations satisfied by the Bessel functions $I_n(x)$.

For $T \to T_c^-$, the coefficients scale as $a_n \sim B^n$ with $B \ll 1$. Therefore, after a transient regime of order $1/Tn^2$, equation (238) can be simplified in

$$a_n = \frac{k}{4T}\frac{1}{n}a_1 a_{n-1}. \tag{240}$$

In particular, for $n = 2$, we get $a_2 = \frac{k}{8T}a_1^2$. This shows that the second mode is slaved to the first. Substituting this in equation (237), we get

$$\xi\frac{da_1}{dt} + (T - T_c)a_1 = -\frac{k^2}{32T}a_1^3. \tag{241}$$

This equation is readily solved (it may be convenient to use $p = a_1^2$ as a variable) and we obtain

$$a_1^2(t) = \frac{A(T_c - T)e^{2(T_c-T)t/\xi}}{1 + \frac{Ak^2}{32T}e^{2(T_c-T)t/\xi}}, \tag{242}$$

where

$$A = \frac{a_1(0)^2}{(T_c - T) - \frac{k^2 a_1(0)^2}{32T}}. \tag{243}$$

For $t \to +\infty$, we get $a_1(\infty)^2 = \frac{32T}{k^2}(T_c - T)$, or equivalently $B^2 = 8T(T_c - T)$. We thus recover the equilibrium result (32) valid close to the critical point. The approach to equilibrium for $t \to +\infty$ is governed by $\delta a_1(t)/a_1(\infty) = \frac{16T}{A^2k^2}e^{-2(T_c-T)t/\xi}$ yielding the damping rate (218). On the other hand, for $t \ll t_{relax}$, one has $a_1(t) = a_1(0)e^{(T_c-T)t/\xi}$ yielding the growth rate (217). Finally, at $T = T_c$, Eq. (241) leads to

$$a_1(t) = \pm\frac{1}{\sqrt{\frac{1}{a_1(0)^2} + \frac{k^2 t}{16\xi T_c}}}. \tag{244}$$

so that the magnetization $B(t) = -\frac{k}{2}a_1(t)$ tends to zero *algebraically* as $t^{-1/2}$ for $t \to +\infty$. Using equation (240), the coefficients $a_n(t)$ are expressed in terms of $a_1(t)$ by

$$a_n = \left(\frac{k}{4T}\right)^{n-1}\frac{1}{n!}a_1^n. \tag{245}$$

In the previous calculations, we have assumed for simplicity that the density profile is symmetrical with respect to the x-axis. The general case is treated in Appendix F. Away from the critical point, the non-local Smoluchowski equation has to be solved numerically. Some numerical simulations are shown in Figures 24–26. Note also that for $T \to 0$ and for sufficiently large times, the density is peaked around $\theta = \pi$ and equation (234) becomes equivalent to the local Kramers equation (227) with $B = kM/2\pi$. Therefore, for $T \to 0$, the eigenvalues of equation (215) tend to $\lambda_n = -nkM/2\pi\xi$ (harmonic oscillator). Substituting $\xi\lambda \to \lambda^2$, we deduce that, for $T \to 0$, the eigenvalues of Eq. (91) tend to $\lambda_n^2 = -nkM/2\pi$ (they are represented in Fig. 9).

7 The multi-species HMF model

We finally briefly comment on the generalization of the preceding results to the case of the multi-species HMF model. We thus return to the Hamiltonian equations (1) and account for the possibility of having particles with different masses. Stellar systems also possess a mass spectrum, so this generalization has a counterpart in astrophysics.

Considering first the statistical equilibrium state, a straightforward generalization of the counting analysis of Section 2.2 yields

$$W(\{n_{ia}\}) = \prod_a N_a! \prod_i \frac{\nu^{n_{ia}}}{n_{ia}!}, \tag{246}$$

for the probability of the state $\{n_{ia}\}$, where n_{ia} gives the number of particles with mass m_a in the ith macrocell. Therefore, the entropy $S = \ln W$ of the multi-species gas is

$$S = -\sum_a \int \frac{f_a}{m_a}\ln\frac{f_a}{m_a}d\theta dv, \tag{247}$$

where $f_a(\theta, v)d\theta dv$ gives the total mass of particles of species a in (θ, v). The distribution function of the whole assembly is

$$f(\theta, v) = \sum_a f_a(\theta, v). \tag{248}$$

The statistical equilibrium state is obtained by maximizing the entropy (247) while conserving the total energy E and the mass M_a of each species of particles. This yields

$$f_a = A'_a e^{-\beta m_a(\frac{v^2}{2}+\Phi)}, \tag{249}$$

which generalizes equation (18). Note that the inverse temperature β is the same for all species in accordance with the theorem of equipartition of energy. This clearly leads to a mass *segregation* since the r.m.s. velocity of species a decreases with the mass: $\langle v^2\rangle_a = T/m_a$. More precisely, equation (249) implies

$$f_a(\epsilon) = C_{ab}[f_b(\epsilon)]^{m_a/m_b}, \tag{250}$$

where C_{ab} is a constant. Therefore, heavy particles will have the tendency to occupy regions of low energy. Recall, by contrast, that there is no mass segregation in Lynden-Bell's statistical theory of violent relaxation for collisionless systems (see Sect. 4) since the mass of the particles does not appear in the Vlasov equation [11]. It would be of interest to study these problems of mass segregation with the HMF model for which numerical simulations are simpler than with gravitational systems.

Considering now the collisional relaxation, a straightforward generalization of the Lenard-Balescu equation to a multi-species system yields

$$\frac{\partial f_a}{\partial t}=\frac{k^2}{4}\frac{\partial}{\partial v}\sum_b\int dv'\frac{\delta(v-v')}{|\epsilon(1,v)|^2}\left(m_b f_b'\frac{\partial f_a}{\partial v}-m_a f_a\frac{\partial f_b'}{\partial v'}\right) \tag{251}$$

with

$$\epsilon(1,\omega)=1+\frac{k}{2}\sum_b\int\frac{f_b'(v)}{v-\omega}dv. \tag{252}$$

We now see that the collision term does not vanish anymore when there are at least two different species. The diffusion and the friction experienced by a particle of one species are caused by collisions with particles of another species. Equation (251) can be rewritten in the suggestive form

$$\frac{\partial f_a}{\partial t}=\frac{\partial}{\partial v}\sum_b\left[D_{ab}\frac{\partial f_a}{\partial v}-\overline{D}_{ab}\frac{\partial f_b}{\partial v}\right], \tag{253}$$

where

$$D_{ab}=\frac{k^2}{4}\frac{m_b f_b}{|\epsilon(1,v)|^2}, \tag{254}$$

is the diffusion coefficient for species a due to collisions with species b and

$$\overline{D}_{ab}=\frac{k^2}{4}\frac{m_a f_a}{|\epsilon(1,v)|^2}, \tag{255}$$

is an "off-diagonal" diffusion coefficient. It corresponds to a friction force

$$\eta_a=-\frac{\overline{D}_{ab}}{f_a}\frac{\partial f_b}{\partial v}. \tag{256}$$

These diffusion coefficients satisfy the relation

$$\frac{\overline{D}_{ab}}{m_a f_a}=\frac{D_{ab}}{m_b f_b}. \tag{257}$$

These results are similar to those obtained by Dubin [48] for the multi-components point vortex gas in two dimensions. When the profile of angular velocity is non-monotonic, the vorticity profile evolves under the effect of long-range collisions caused by a process of resonance [47, 45,3]. When the profile of angular velocity is monotonic, there is no evolution for the single-species system. An evolution is, however, possible for the distribution function of each species in the multi-components system. In a sense, at the order $1/N$, the kinetic theory of the point vortex gas (evolution of the single-species system when the angular velocity profile is non-monotonic) is intermediate between the kinetic theory of stellar systems (evolution of the single-species system in any case) and the kinetic theory of the HMF model (no evolution of the single-species system).

Therefore, the results of Dubin [48] can be directly transposed to the present context. In particular, we note that the total distribution function $\sum_a f_a(v,t)=f(v)$ is stationary so that the conservation of energy is trivially satisfied. On the other hand, a H-theorem can be proved for the entropy (247), i.e. $\dot{S}\geq 0$. The equality $\dot{S}=0$ corresponds to vanishing currents

$$J_a=-\sum_b\left[D_{ab}\frac{\partial f_a}{\partial v}-\overline{D}_{ab}\frac{\partial f_b}{\partial v}\right]=0, \tag{258}$$

implying the following equilibrium relation between the densities

$$f_a(v)=K_{ab}[f_b(v)]^{m_a/m_b}, \tag{259}$$

where K_{ab} is a constant independent on v. This equation is similar to equation (250) but, here, $f_a(v)$ is not necessarily the Maxwellian. Indeed, equation (259) is satisfied by any distribution of the form $f_a(v)=A_a\exp[-\beta m_a\chi(v)]$, where $\chi(v)$ is determined by the initial conditions.

We can use these results to study the relaxation of a test particle of mass m in a bath of field particles with mass m_f. Neglecting collective effects for simplicity, we find that the equivalent of the Fokker-Planck equation (184) is now

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial v}\left[D(v)\left(\frac{\partial P}{\partial v}+\beta m P v\right)\right]. \tag{260}$$

with

$$D(v)=\frac{nk^2}{4}m_f^2\left(\frac{\beta m_f}{2\pi}\right)^{1/2}e^{-\beta m_f\frac{v^2}{2}}, \tag{261}$$

where n is the number density of field particles. The equilibrium distribution of the test particle is

$$P_{eq}(v)=\left(\frac{\beta m}{2\pi}\right)^{1/2}e^{-\beta m\frac{v^2}{2}}. \tag{262}$$

The timescale of collisional relaxation is

$$t_r=\frac{v_{mf}^3}{0.121 n m_f^2 k^2}, \tag{263}$$

and $t_r'=2(m_f/m)t_r$. In dimensionless form, equation (260) can be rewritten

$$\frac{\partial P}{\partial t}=\frac{1}{t_R}\frac{\partial}{\partial w}\left[G(w)\left(\frac{\partial P}{\partial w}+2\frac{m}{m_f}Pw\right)\right], \tag{264}$$

with

$$t_R=\frac{v_m^3}{0.05 n m_f^2 k^2} \tag{265}$$

and $G(w) = e^{-w^2}$. If we properly account for collective effects, the diffusion coefficient is given by

$$D(v) = \frac{\frac{nk^2}{4} m_f^2 (\frac{\beta m_f}{2\pi})^{1/2} e^{-\beta m_f \frac{v^2}{2}}}{[1 - \eta A(\sqrt{\beta m_f} v)]^2 + \frac{\pi}{2}\eta^2 \beta m_f v^2 e^{-\beta m_f v^2}}, \quad (266)$$

where $\eta = \frac{kMm_f}{4\pi T}$. On the other hand, for a distribution of the bath of the form $f_0(v)$, the Fokker-Planck equation is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left[D(v)\left(\frac{\partial P}{\partial v} - \frac{m}{m_f} P \frac{d\ln f_0}{dv}\right)\right], \quad (267)$$

with $D(v) = \frac{k^2}{4} m_f f_0(v)/|\epsilon(1,v)|^2$. We note that the equilibrium distribution of the test particle is $P_{eq}(v) = Af_0(v)^{m/m_f}$.

8 Conclusion

In this paper, we have given an exhaustive description of the HMF model that recently appeared in statistical mechanics as a simple model with long-range interactions similar to self-gravitating systems. The originality of our approach is to offer an overview of the subject and to see how different models (Hamiltonian, Brownian, fluids,...) are related to each other. Other studies concentrate in general on a specific aspect of the problem. We think that putting all the models in parallel is illuminating because they are closely connected to each other so that a unified (and aesthetic) description can be given. These connections were previously noted by one of us (P.H.C) in the case of 3D self-gravitating systems and it was natural to extend these results to the HMF model. A more general approach is given in Chavanis [17] for an arbitrary potential of interaction in D dimensions. The present paper can be seen as a particular application of this general formalism for a one dimensional potential truncated to one mode. The main interest of the HMF model in this context is to yield simple explicit results.

Another originality of our approach is to emphasize the connection between the HMF model and self-gravitating systems (and 2D vortices) although this link is only sketched in other papers, except in the early work of Inagaki. Many concepts and technics that are well-known in astrophysics have been rediscovered for the HMF model, sometimes with a different point of view. This is true in particular for the notion of violent relaxation and metaequilibrium states. In statistical mechanics, this has been approached via a notion of "generalized thermodynamics" (Tsallis [36]) although it was understood early in astrophysics (Lynden-Bell [11], Tremaine et al. [14]) that these metaequilibrium states correspond to particular stationary solutions of the Vlasov equations on a coarse-grained scale (Chavanis et al. [39], Chavanis [19]). Thus, our dynamical interpretation of Tsallis functional as a particular H-function differs from the thermodynamical interpretation given by Boghosian [60], Latora et al. [61] and Taruya & Sakagami [62].

Concerning the interest of the HMF model for astrophysicists, we have shown that it exhibits the same types of behaviors as 3D self-gravitating systems while being much simpler to study because it is one dimensional and avoids complicated problems posed by the divergence of the gravitational potential at short distances and the absence of a large-scale confinement. Thus, it distinguishes what is common to long-range interactions and what is specific to gravity. This comparative study should bring new light in the statistical mechanics of self-gravitating systems which has long been a controversial subject. Other simplified models of gravity have been introduced such as the parallel planar sheets of Camm [63], the concentric spherical shells of Hénon [64] or the toy models of Lynden-Bell & Lynden-Bell [65] and Padmanabhan [2]. These toy models have often allowed advances in the description of more realistic self-gravitating systems that are difficult to study in full detail. We think that, similarly, the HMF model should find its place in the astrophysical literature.

Another interest of the HMF model is to allow to study in great detail what happens close to the critical point. In the HMF model, the potential is truncated to one mode $n = 1$ and there exists a critical temperature $T_c = kM/4\pi$ below which the uniform phase is unstable. For infinite homogeneous self-gravitating systems, there is a continuous spectrum of modes but there exists a critical wavelength $\lambda_J = (\pi T/Gm\rho)^{1/2}$ (depending on the temperature) above which the system is unstable. Alternatively, if we fix the size of the system (for example by placing it in a box of radius R), the maximum wavelength is R and the Jeans instability criterion now determines a critical temperature $T_c \sim GmM/(\pi R)$ below which the system becomes unstable (we have used $\lambda_J = R$ and $\rho \sim M/R^3$). In fact, in the case of box-confined gravitational systems, we must consider that the gaseous phase is inhomogeneous and use a more precise stability analysis (Chavanis [20]) yielding $T_c = GmM/(2.52R)$. These remarks show that the critical temperature in the HMF model plays exactly the same role as the critical temperature in finite isothermal spheres. Now, in the framework of the HMF model, it is possible to study how the mean-field results are altered close to the critical point due to collective effects. This is more complicated for self-gravitating systems because they are inhomogeneous. However, on a qualitative point of view, we expect similar results: divergence of the two-point correlation function like in equation (61), divergence of the force auto-correlation function like in equation (64), alteration of the diffusion coefficient and increase of the relaxation time like in equation (190), increase of the decorrelation time like in equation (197)... These problems are difficult to study for self-gravitating systems but they are of considerable importance. They have never been discussed in detail because it is usually implicitly assumed that the system size is much smaller than the Jeans scale (or the temperature much larger than T_c) so that collective effects are neglected and the dielectric function is approximated as $\epsilon \simeq 1$. The present simplified study is a first step to understand the failure of the mean-field approximation close to the critical point and it can thus find important applications in theoretical astrophysics. Note that fluctuations in isothermal spheres close to the critical point have been studied by Katz and Okamoto [66] and Chavanis [18].

Finally, we have given an astrophysical application of the HMF model in relation with the formation of bars in spiral galaxies, following the original idea of Pichon and Lynden-Bell. This simple model is consistent with the phenomenology of structure formation which results from a competition between long-range interactions (gravity) and thermal motion (velocity dispersion). It is sometimes argued that the moment of inertia α^{-1} of stellar orbits can be negative [10]. In that case, we must consider equation (1) with $k < 0$. This corresponds to the repulsive (anti-ferromagnetic) HMF model. Now, Barré et al. [30] have observed that this model leads to the formation of a bicluster. In the stellar disk analogy, the equivalent of the "bicluster" would be two bars perpendicular to each other. We do not know whether this type of structure is observed in astrophysics.

One of us (P.H.C) is grateful to Donald Lynden-Bell for inspiring discussions some years ago. He is also grateful to Pr. D. Dubin for drawing his attention to his work on non-neutral plasmas. We would like to thank T. Dauxois for interesting discussions.

Appendix A: Estimate of the eigenvalue close to the critical point

For simplicity, we restrict ourselves to symmetrical perturbations with respect to the x-axis. The general case is treated in Appendix F by another method. In the canonical ensemble, we have to study the eigenvalue equation ($V = 0$):

$$\frac{d}{d\theta}\left(\frac{1}{\rho(\theta)}\frac{dq}{d\theta}\right)+\frac{k}{2\pi T}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = 2\lambda q, \tag{268}$$

where $\rho(\theta)$ is the equilibrium density profile and $q(\theta)$ is the perturbation. We consider the clustered phase close to the critical point T_c. Thus, we can perform a systematic expansion of the parameters in powers of $B \ll 1$ or, equivalently, in powers of $x = \beta B \ll 1$.

Using equations (20) and (21), the density profile can be expanded as

$$\frac{1}{\rho} = \frac{2\pi}{M}\left[1 + x\cos\theta + \frac{x^2}{4}(1+2\cos^2\theta) + ...\right]. \tag{269}$$

Substituting this result in equation (268), and using the expansion (30) of the temperature, we obtain

$$\frac{d}{d\theta}\left\{\pi\left[1 + x\cos\theta + \frac{x^2}{4}(1+2\cos^2\theta)\right]\frac{dq}{d\theta}\right\} + \left(1+\frac{x^2}{8}\right)\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = \mu x^2 q. \tag{270}$$

where we have set $\lambda M = \mu x^2$ with $\mu = O(1)$. Here, λ refers to the largest eigenvalue of equation (268) which is equal to zero for $x = 0$ (the other eigenvalues are $\lambda_n M = -\pi n^2 + O(x)$ for $n \geq 2$). Furthermore, the following expansion shows that the term of order x vanishes so we have directly written $\lambda \sim x^2$. According to equation (30), we have

$$\lambda = \frac{8\mu}{M}\left(\frac{\beta}{\beta_c} - 1\right), \tag{271}$$

where μ has to be determined self-consistently. We thus expand the perturbation as

$$q(\theta) = q_0(\theta) + x\ q_1(\theta) + x^2\ q_2(\theta) + ... \tag{272}$$

and we introduce the differential operator

$$\mathcal{L}q = \pi\frac{d^2q}{d\theta^2} + \int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta'. \tag{273}$$

To order 0, we have

$$\mathcal{L}q_0 = 0, \tag{274}$$

yielding $q_0 = \sin\theta$. To order 1, we get

$$\mathcal{L}q_1 = \pi\sin(2\theta), \tag{275}$$

yielding

$$q_1 = -\frac{1}{4}\sin(2\theta) + C\sin\theta, \tag{276}$$

where C is an arbitrary constant. Finally, to order 2, we have after simplification

$$\mathcal{L}q_2 = \left(\mu + \frac{\pi}{4}\right)\sin\theta + \pi C\sin(2\theta) - \frac{3\pi}{8}\sin(3\theta), \tag{277}$$

yielding

$$\mu = -\frac{\pi}{4}, \tag{278}$$

and

$$q_2 = D\sin\theta - \frac{C}{4}\sin(2\theta) + \frac{1}{24}\sin(3\theta), \tag{279}$$

where D is an arbitrary constant. Therefore, close to the critical point, the largest eigenvalue of equation (268) is

$$\lambda M = -\frac{\pi}{4}x^2, \qquad \text{or} \qquad \lambda M = -2\pi\left(\frac{\beta}{\beta_c} - 1\right). \tag{280}$$

We can obtain the expression of the eigenvalue by a slightly different method. We consider the Hilbert space of 2π-periodic continuous real functions with scalar product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)g(\theta)d\theta. \tag{281}$$

We note that the operator (273) is Hermitian in the sense that

$$\langle \mathcal{L}f, g\rangle = \langle f, \mathcal{L}g\rangle. \tag{282}$$

The equation obtained to second order can be written

$$\mathcal{L}q_2 = g(\theta), \tag{283}$$

with

$$g(\theta) = \left(\mu + \frac{\pi}{4}\right)\sin\theta + \pi C\sin(2\theta) - \frac{3\pi}{8}\sin(3\theta). \tag{284}$$

We note q_0 the Kernel of $\mathcal{L}$, i.e. $\mathcal{L}q_0 = 0$. Then, we have the condition of solvability

$$\langle q_0, g\rangle = 0. \tag{285}$$

Indeed,

$$\langle q_0, g\rangle = \langle q_0, \mathcal{L}q_2\rangle = \langle \mathcal{L}q_0, q_2\rangle = 0. \tag{286}$$

In our case, $q_0 = \sin\theta$, so that the condition $\langle q_0, g\rangle = 0$ with (284) yields equation (278).

In the microcanonical ensemble, we have to study the eigenvalue equation ($V \neq 0$):

$$\frac{d}{d\theta}\left(\frac{1}{\rho(\theta)}\frac{dq}{d\theta}\right) + \frac{k}{2\pi T}\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = \frac{2V}{MT^2}\frac{d\Phi}{d\theta} + 2\lambda' q. \tag{287}$$

To leading order in $B \ll 1$, the term $\frac{2V}{MT^2}\frac{d\Phi}{d\theta}$ can be written $\frac{2\pi}{M}x^2\sin\theta$. Therefore, equation (287) becomes

$$\frac{d}{d\theta}\left\{\pi\left[1 + x\cos\theta + \frac{x^2}{4}(1+2\cos^2\theta)\right]\frac{dq}{d\theta}\right\} + \left(1+\frac{x^2}{8}\right)\int_0^{2\pi} q(\theta')\cos(\theta-\theta')d\theta' = \pi x^2\sin\theta + \mu' x^2 q. \tag{288}$$

The eigenvalue is now

$$\mu' = -\frac{\pi}{4} - \pi = -\frac{5\pi}{4}, \tag{289}$$

yielding

$$\lambda M = -\frac{5\pi}{4}x^2, \qquad \text{or} \qquad \lambda M = -10\pi\left(\frac{\beta}{\beta_c} - 1\right). \tag{290}$$

Although the onset of instability is the same in the two ensembles, the eigenvalues differ in the condensed phase.

Appendix B: Some useful identities

For any system described by a distribution function $f(\theta, v)$, we define the density and the pressure by

$$\rho = \int f dv, \qquad p = \int f v^2\, dv. \tag{291}$$

The kinetic temperature $T = p/\rho$ is equal to the velocity dispersion of the particles. If the distribution function is of the form $f = f(\epsilon)$ with $\epsilon = \frac{v^2}{2} + \Phi(\theta)$, we have

$$\frac{dp}{d\theta} = \int f'(\epsilon)\frac{d\Phi}{d\theta}v^2 dv = \frac{d\Phi}{d\theta}\int \frac{\partial f}{\partial v} v dv = -\frac{d\Phi}{d\theta}\int f dv = -\rho\frac{d\Phi}{d\theta}, \tag{292}$$

which returns the condition of hydrostatic balance (70). This relation is equivalent to

$$\frac{p'(\rho)}{\rho} = -\frac{1}{\rho'(\Phi)}. \tag{293}$$

Now, we note that

$$\frac{d\rho}{d\Phi} = \int f'(\epsilon)\ dv = \int \frac{\partial f}{\partial v}\frac{1}{v}\ dv. \tag{294}$$

This yields the important identity

$$\frac{p'(\rho)}{\rho} = \frac{-1}{\int \frac{\partial f}{\partial v}\frac{1}{v}\ dv}. \tag{295}$$

This identity is valid for both homogeneous and inhomogeneous systems. It may be useful to rederive it in the case of homogeneous systems since $\Phi = 0$ in that case so that the above procedure is ill-determined.

We consider a homogeneous system described by the distribution function $f = F(\beta\frac{v^2}{2} + \alpha)$ where α is a Lagrange multiplier taking into account the conservation of mass (normalization). In that case, $\rho = \rho(\alpha)$ and $p = p(\alpha)$. Now,

$$\frac{dp}{d\alpha} = \int F'(\beta\frac{v^2}{2} + \alpha)v^2\ dv = \int \frac{\partial}{\partial v}F(\beta\frac{v^2}{2}+\alpha)\frac{v}{\beta}dv = -\frac{\rho}{\beta}, \tag{296}$$

and

$$\frac{d\rho}{d\alpha} = \int F'(\beta\frac{v^2}{2}+\alpha)\ dv = \int \frac{\partial}{\partial v}F(\beta\frac{v^2}{2}+\alpha)\frac{1}{\beta v}dv = \int \frac{f'(v)}{\beta v}dv. \tag{297}$$

Eliminating α from the foregoing relations, we obtain

$$\frac{p'(\rho)}{\rho} = \frac{-1}{\int \frac{f'(v)}{v}dv}, \tag{298}$$

which is consistent with equation (295). Now, introducing the velocity of sound $c_s^2 = p'(\rho)$ and using $\rho = M/2\pi$, we get

$$c_s^2 = -\frac{M}{2\pi}\frac{1}{\int_{-\infty}^{+\infty}\frac{f'(v)}{v}dv}. \tag{299}$$

Appendix C: General dispersion relation for a gaseous system

We consider the Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{300}$$

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p - \rho\nabla\Phi, \tag{301}$$

in D dimensions and for an arbitrary binary potential of interaction $u(|\mathbf{r}-\mathbf{r}'|)$ such that

$$\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}',t)d^D\mathbf{r}'. \tag{302}$$

We also consider an arbitrary barotropic equation of state $p = p(\rho)$. Clearly, $\rho = \text{Cst.}$, $\mathbf{u} = \mathbf{0}$ and $\Phi = 0$ is a stationary solution of equation (300–302) provided that $\int u(\mathbf{x})d^D\mathbf{x} = 0$ (for the gravitational potential, this is not the case and we must advocate the Jeans swindle). We shall restrict ourselves to such homogeneous solutions. The linearized Euler equations are

$$\frac{\partial \delta\rho}{\partial t} + \rho\nabla\cdot\delta\mathbf{u} = 0, \tag{303}$$

$$\rho\frac{\partial \delta\mathbf{u}}{\partial t} = -p'(\rho)\nabla\delta\rho - \rho\nabla\delta\Phi, \tag{304}$$

$$\delta\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r}',t)d^D\mathbf{r}'. \tag{305}$$

These equations can be combined to give

$$\frac{\partial^2\delta\rho}{\partial t^2} - c_s^2\Delta\delta\rho - \rho\Delta\delta\Phi = 0, \tag{306}$$

where we have introduced the velocity of sound $c_s^2 = p'(\rho)$. We introduce the Fourier transform of the interaction potential such that

$$u(\mathbf{x}) = \int e^{i\mathbf{k}\cdot\mathbf{x}}\hat{u}(\mathbf{k})d^D\mathbf{k}. \tag{307}$$

Taking the Fourier transform of equations (305) and (306) with the convention $\delta\rho \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, and combining the resulting expressions, we find that

$$\omega^2 = c_s^2k^2 + (2\pi)^D\hat{u}(\mathbf{k})k^2\rho, \tag{308}$$

which is the required dispersion relation. The system will be unstable if

$$c_s^2 + (2\pi)^D\hat{u}(\mathbf{k})\rho < 0. \tag{309}$$

In particular, it is necessary that $\hat{u}(\mathbf{k}) < 0$ corresponding to attractive potentials. In that case, the condition of instability reads

$$c_s^2 < (c_s^2)_{crit} \equiv (2\pi)^D\rho|\hat{u}(\mathbf{k})|_{max}. \tag{310}$$

Then, the unstable lengthscales are determined by equation (309). Various situations can happen [17] depending on the form of the potential $\hat{u}(\mathbf{k})$. For the gravitational interaction in $D = 3$, using $\Delta u = 4\pi G\delta(\mathbf{x})$, we have $(2\pi)^3\hat{u}(\mathbf{k}) = -4\pi G/k^2$. We recover the usual formula [15]:

$$\omega^2 = c_s^2k^2 - 4\pi G\rho. \tag{311}$$

The system is always unstable ($(c_s^2)_{crit} = \infty$) for sufficiently large wavelengths or equivalently for

$$k < k_J \equiv \left(\frac{4\pi G\rho}{c_s^2}\right)^{1/2}, \tag{312}$$

where k_J is the Jeans wave number. For the HMF model, $\hat{u}_n = 0$ if $n \neq \pm 1$ and $\hat{u}_{\pm 1} = -\frac{k}{4\pi}$. In that case, equation (308) returns equation (87). The system is unstable for the mode $n = 1$ if $c_s^2 < kM/4\pi$.

Appendix D: General dispersion relation for a stellar system

We consider the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0, \tag{313}$$

in D dimensions and for an arbitrary binary potential of interaction $u(|\mathbf{r}-\mathbf{r}'|)$ as before. Clearly, a distribution function $f = f_0(\mathbf{v})$ which depends only on the velocity is a stationary solution of equation (313) under the same assumptions as before. We shall restrict ourselves to such homogeneous solutions. The linearized Vlasov equation is

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f}{\partial \mathbf{r}} - \nabla\delta\Phi\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0, \tag{314}$$

which must be supplemented by

$$\delta\Phi(\mathbf{r},t) = \int u(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r}',t)d^D\mathbf{r}'. \tag{315}$$

Taking the Fourier transform of equations (314) and (315) with the convention $\delta f \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ and combining the resulting expressions, we find that

$$(\omega - \mathbf{k}\cdot\mathbf{v})\delta\hat{f} + (2\pi)^D\hat{u}(\mathbf{k})\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}\int\delta\hat{f}(\mathbf{k},\mathbf{v},\omega)d^D\mathbf{v} = 0. \tag{316}$$

This can be rewritten

$$\epsilon(\mathbf{k},\omega) \equiv 1 + (2\pi)^D\hat{u}(\mathbf{k})\int\frac{\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}}d^D\mathbf{v} = 0, \tag{317}$$

which is the dispersion relation of the system. For the gravitational interaction in $D = 3$, using $(2\pi)^3\hat{u}(\mathbf{k}) = -4\pi G/k^2$, we recover the usual formula [15]:

$$1 - \frac{4\pi G}{k^2}\int \frac{\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} d^D\mathbf{v} = 0. \tag{318}$$

For the HMF model, we recover equation (117).

Appendix E: General dispersion relation for a Brownian system

We finally consider a gas of Brownian particles in interaction described in the strong friction limit by the Smoluchowski equation

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\left[\frac{1}{\xi}(T\nabla\rho + \rho\nabla\Phi)\right], \tag{319}$$

where Φ is given by equation (302). Under the same conditions as in Appendix C, the linearized Smoluchowski equation is

$$\frac{\partial\delta\rho}{\partial t} = \nabla\cdot\left[\frac{1}{\xi}(T\nabla\delta\rho + \rho\nabla\delta\Phi)\right]. \tag{320}$$

Taking the Fourier transform of equation (320), we obtain the dispersion relation

$$i\xi\omega = Tk^2 + (2\pi)^D\hat{u}(k)k^2\rho, \tag{321}$$

which can be compared to equation (308). In the gravitational case, we get

$$i\xi\omega = Tk^2 - 4\pi G\rho. \tag{322}$$

The stability condition coincides with the Jeans criterion for an isothermal gas

$$k < k_J \equiv \left(\frac{4\pi G\rho}{T}\right)^{1/2}. \tag{323}$$

However, the stable modes are exponentially damped in the case of Brownian particles while they have an oscillatory nature in the case of gaseous systems. For a sinusoidal potential of interaction, we recover equation (217).

Appendix F: Generalization of the analytical solution (242) to arbitrary perturbations

In Section 6.4, we have restricted our analysis to density profiles that are always even, i.e. $\rho(-\theta,t) = \rho(\theta,t)$. It is not difficult to relax this hypothesis. Let us write the density profile in the general form

$$\rho = \sum_{n=-\infty}^{+\infty} a_n(t)e^{in\theta}. \tag{324}$$

Substituting this decomposition in equation (234), we find that

$$a_0 = \frac{M}{2\pi}, \tag{325}$$

$$\xi\frac{da_n}{dt} + Tn^2 a_n = \frac{k}{2}n(a_1 a_{n-1} - a_{-1}a_{n+1}). \tag{326}$$

To first order in $T_c - T \ll T_c$, we can neglect a_n with $|n| \geq 3$. We thus get

$$a_{\pm 2} = \frac{k}{4T}a_{\pm 1}^2. \tag{327}$$

The equations for the modes $a_{\pm 1}$ are therefore

$$\xi\frac{da_1}{dt} + (T - T_c)a_1 = -\frac{k^2}{8T}a_1^2 a_{-1}, \tag{328}$$

$$\xi\frac{da_{-1}}{dt} + (T - T_c)a_{-1} = -\frac{k^2}{8T}a_{-1}^2 a_1. \tag{329}$$

At that point, it is convenient to introduce the variables $p = a_1 a_{-1}$ and $X = a_1 + a_{-1}$. They satisfy the differential equations

$$\xi\frac{dX}{dt} + (T - T_c)X = -\frac{k^2}{8T}pX, \tag{330}$$

$$\xi\frac{dp}{dt} + 2(T - T_c)p = -\frac{k^2}{4T}p^2. \tag{331}$$

The equation for p is readily solved and we obtain

$$p(t) = \frac{2A(T_c - T)e^{2(T_c - T)t/\xi}}{1 + \frac{Ak^2}{4T}e^{2(T_c - T)t/\xi}}. \tag{332}$$

Substituting this result in equation (330) and solving the resulting equation, we get

$$X(t) = \frac{B}{\sqrt{|\frac{Ak^2}{4T} + e^{-2(T_c - T)t/\xi}|}}. \tag{333}$$

The modes a_1 and a_{-1} are deduced from X and p by solving the second order equation $a^2 - Xa + p = 0$ yielding

$$a_{\pm 1}(t) = \frac{X(t) \pm \sqrt{\Delta(t)}}{2}. \tag{334}$$

The discriminant can be written

$$\Delta = \frac{B^2 - 8|A|(T_c - T)}{|\frac{k^2 A}{4T} + e^{-2(T_c-T)t/\xi}|}. \qquad (335)$$

The constants of integration A and B, which can be positive or negative, are fixed by the initial condition. They must satisfy $\Delta(0) \geq 0$, i.e. $B^2 \geq 8|A|(T_c - T)$. Then, $\Delta(t) \geq 0$ at all times. If initially $a_{-1}(0) = a_1(0)$, then $\Delta(0) = 0$. By equation (335), this implies that $\Delta(t) = 0$ for all times. Therefore, if the initial perturbation is even, it remains even during all the evolution. We thus recover the results of Section 6.4 with a_1 equals to $2a_{\pm 1}$ with the present notations. Finally, for $T = T_c$, equations (330) and (331) lead to

$$p(t) = \frac{1}{\frac{1}{p(0)} + \frac{k^2 t}{4\xi T_c}}, \quad X(t) = \frac{X(0)}{\sqrt{1 + \frac{k^2 p(0) t}{4\xi T_c}}}. \qquad (336)$$

We can also use this approach to study the dynamical stability of stationary solutions of equation (234). By truncation to order 2 ($a_3 = a_{-3} = 0$), we obtain the equilibrium relations: $B = -k a_1^e$ (we suppose $a_1^e = a_{-1}^e$ without lack of generality), $a_2^e = a_{-2}^e = \frac{2}{k}(T_c - T)$ and $B^2 = 8T_c(T_c - T)$. These expressions are valid up to order 1 in $T_c - T$. Let us compute the four smallest eigenvalues corresponding to the relaxation towards equilibrium. Defining $a_n = a_n^e + \exp(\lambda t)\delta a_n$, we obtain after linearization $MY = 0$ where Y is a row vector of components ${}^T(\delta a_1, \delta a_{-1}, \delta a_2, \delta a_{-2})$ and

$$M = \begin{pmatrix} -\xi\lambda + \Delta T & -\Delta T & -\sqrt{2T_c \Delta T} & 0 \\ -\Delta T & -\xi\lambda + \Delta T & 0 & -\sqrt{2T_c \Delta T} \\ 4\sqrt{2T_c \Delta T} & 0 & -\xi\lambda - 4T & 0 \\ 0 & -4\sqrt{2T_c \Delta T} & 0 & -\xi\lambda - 4T \end{pmatrix}$$

where we have set $\Delta T = T_c - T$. The eigenvalues are the zeros of the determinant of M. The two lowest eigenvalues scale as $(T_c - T)$ so we set $\lambda = \frac{1}{\xi}(T_c - T)\lambda_1$. First dividing the two first rows and the two first columns by $\sqrt{(T_c - T)}$, and dividing the two last rows by $2\sqrt{2T_c}$ and the two last columns by $\sqrt{2T_c}$ we obtain at leading order

$$\det \begin{pmatrix} -\lambda_1 + 1 & -1 & -1 & 0 \\ -1 & -\lambda_1 + 1 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix} = 0$$

which gives $\lambda_1^2 + 2\lambda_1 = 0$. The smallest eigenvalue $\lambda_1 = 0$ is the neutral mode associated to the angle rotation invariance, whereas the smallest non zero eigenvalue is $\lambda_1 = -2$ or equivalently $\lambda = -\frac{2}{\xi}(T_c - T)$. The two consecutive eigenvalues are of order 0. Considering the determinant of M at order 0, we obtain two degenerate eigenvalues $\lambda = -4T_c/\xi$. We note that this degeneracy will be removed at next order.

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